What about resurgence?

(An introduction to the introduction)

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Conclusion

1. There is non-perturbative data encoded in perturbative series.
2. Resurgence (theory and Alien Calculus) can decipher this data and enhance perturbation theory.

Perturbation theory is a way to reach non-perturbative theory as they are linked.

A few introductory references

- D. Dorigoni. "An Introduction to Resurgence, Trans-Series and Alien Calculus". In: *Annals Phys.* 409 (2019), p. 167914. arXiv: `1411.3585 [hep-th]`
- D. Sauzin. "Introduction to 1-summability and resurgence". 2014. arXiv: `1405.0356 [math.DS]`
- I. Aniceto, G. Basar, and R. Schiappa. "A Primer on Resurgent Transseries and Their Asymptotics". In: *Phys. Rept.* 809 (2019), pp. 1–135. arXiv: `1802.10441 [hep-th]`
- M. Mariño. "Lectures on non-perturbative effects in large N gauge theories, matrix models and strings". In: *Fortsch. Phys.* 62 (2014), pp. 455–540. arXiv: `1206.6272 [hep-th]`
- G. V. Dunne and M. Ünsal. "What is QFT? Resurgent trans-series, Lefschetz thimbles, and new exact saddles". In: *PoS* LATTICE2015 (2016), p. 010. arXiv: `1511.05977 [hep-lat]`

A generic phenomena

We seek $f(x)$ for which we have no explicit solution.

For x “small” we expand $f(x)$ using

- Rayleigh–Schrödinger perturbation theory,
- (Uniform)-WKB,
- Kato’s method,
- saddle point/steepest descent, ...

to get:

$$\widetilde{f}(x) = a_0 + a_1 x + \ldots = \sum_{n \geqslant 0} a_n x^n,$$

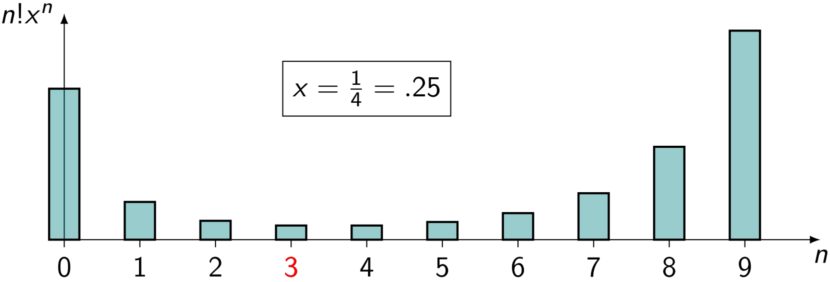
in which each added term is a ‘smaller correction’. But $\widetilde{f}(x)$ is divergent!

$\widetilde{f}$ doesn’t exist as a function, it is formal, a handy way to store the perturbative data a_1, a_2, ...

Euler's example

$$\widetilde{f}(x) = \sum_{n \geqslant 0} (-1)^n n! x^{n+1}$$

is the (divergent) perturbative solution to $x^2 f'(x) + f(x) = x$.



Terms first decrease to a 'least term' then blow up. (Here $n = 3$.)

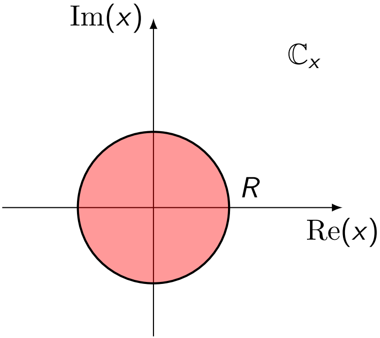
Dyson's argument

The (formal) series $\widetilde{f}(x) = \sum_{n=0} a_n x^n$ being divergent is GOOD.

Often the situation is **polarized** with:

- stable theory for $x \geqslant 0$;
- unstable theory for $x < 0$.

If $\widetilde{f}(x)$ *was* convergent in a disk of radius R go from $x \geqslant 0$ to $x < 0$ without trouble $\Rightarrow R = 0$: $\widetilde{f}$ *has to be divergent.*



Bonus: x needs not to be "small" anymore since it's always too big.

F. J. Dyson. "Divergence of perturbation theory in quantum electrodynamics". In: *Phys. Rev.* 85 (1952), pp. 631–632.

A nice stance

Il y a entre les géomètres et les astronomes une sorte de malentendu au sujet de la signification du mot convergence. Les géomètres (...) disent qu'une série est convergente quand la somme des termes tend vers une limite déterminée, quand même les premiers termes diminueraient très lentement. Les astronomes, au contraire, ont coutume de dire qu'une série converge quand les vingt premiers termes, par exemple, diminuent très rapidement, quand même les termes suivants devraient croître indéfiniment.

There is a sort of misunderstanding between geometers and astronomers about the meaning of the word convergence. Geometers (...) say that a series is convergent when the sum of the terms tends toward a determined limit, even if the first terms diminish very slowly. Astronomers, on the contrary, are accustomed to saying that a series converges when the first twenty terms, for example, diminish very rapidly, even if the following terms should increase indefinitely.

H. Poincaré. *Les méthodes nouvelles de la mécanique céleste*. Vol. 2. 1893.

A nice stance

Les deux règles sont légitimes : la première, dans les recherches théoriques; la seconde, dans les applications numériques. Toutes deux doivent régner, mais dans deux domaines séparés et dont il importe de bien connaître les frontières. Les astronomes ne les connaissent pas toujours d'une façon bien précise, mais ils les franchissent rarement; l'approximation dont ils se contentent les maintient d'ordinaire beaucoup en deçà; d'ailleurs leur instinct les guide et, s'il les trompait, le contrôle de l'observation les avertirait promptement de leur erreur.

Both rules are legitimate: the first, in theoretical research; the second, in numerical applications. Both must reign, but in two separate domains, the boundaries of which it is important to know well. Astronomers do not always know them in a very precise way, but they rarely cross them; the approximation with which they are content usually keeps them well short of them; moreover, their instinct guides them and, if it deceives them, the control of observation would promptly warn them of their error.

A nice stance

Or in the words of K.O. Friedrichs:

> "However this may be, if a divergent series is useful it must be meaningful."
>
> and in particular:
>
> "formal procedures –I mean those used by good physicists– indeed are valid if only the meaning of validity is properly interpreted."

K.O. Friedrichs. "Asymptotic phenomena in mathematical physics". In: *Bull. Am. Math. Soc.* 61.6 (1955), pp. 485–504.

Asymptotic expansion

Definition

$\widetilde{f}(x)$ is *asymptotic* to $f(x)$ if: $\lim_{x\to 0} x^{-N}\left|f(x)-\sum_{n=0}^{N} a_n x^n\right| = 0$.

The remainder $R_N(x)$ does not necessarily go to 0 for $N\to\infty$.

Generically perturbative expansions are asymptotic expansions whose large order behavior is:

$$a_n \sim K A^{-(n+\mu)}\Gamma(n+\mu)\left(1+O\left(\frac{1}{n}\right)\right), \quad n\to\infty,$$

with K, A and μ constants specific to the problem.

⚠ Two different functions can have the same asymptotic expansion.

J. Zinn-Justin. “Perturbation Series at Large Orders in Quantum Mechanics and Field Theories: Application to the Problem of Resummation”. In: *Phys. Rept.* 70 (1981), p. 109.

Optimal truncation (Astronomers' sums)

Stopping the sum at the least term minimizes the error term:

$$n_{opt}(x) = \left\lfloor \left| \frac{A}{x} \right| \right\rfloor$$

provides a numerical approximation of f, but gets choppy for x 'large'.

Example : Euler's case

$$\widetilde{f}(.25) = 0.1953125 \qquad \text{(stopped at } n_{opt} - 1 = 3\text{)},$$
$$f(.25) = 0.20634564990105583 31... = -e^4 \mathrm{Ei}(-4).$$

This is what we do in QED with $\alpha \approx \frac{1}{137}$ and $n_{opt} \approx 100$.

M. Flory, R. C. Helling, and C. Sluka. "How I Learned to Stop Worrying and Love QFT". In: (2012). arXiv: `1201.2714 [math-ph]`.

Borel-Laplace resummation: Borel transform

We seek a summation process $\mathcal{S}$ to apply to $\widetilde{f}$ providing a better result.

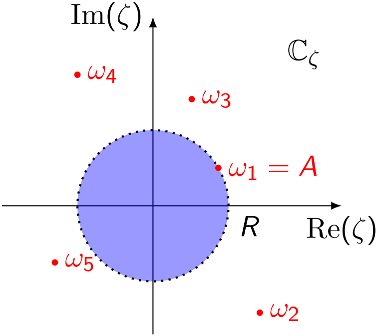
To $\widetilde{f}(x)$ associate its Borel transform:

$$\widehat{f}(\zeta) = \mathcal{B}[\widetilde{f}](\zeta) = \sum \frac{a_n}{n!}\zeta^n.$$

The factorial growth is killed and the series is convergent in the Borel plane in a disk of radius

$$R = \min d(\omega_i, 0) = |A|.$$

$\omega_i \equiv$ singularities of $\widehat{f}(\zeta)$, $\Omega = \{\omega_i\}$
$\Omega \sim$ instantons, renormalons

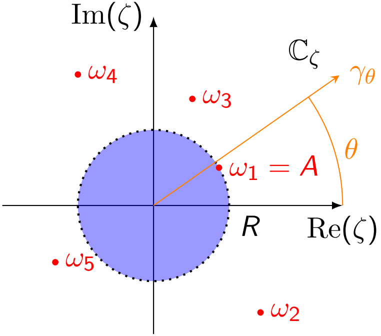


Borel-Laplace resummation: directional Laplace transform

In $\mathbb{C}_\zeta$ set a direction $\gamma_\theta = e^{i\theta}\mathbb{R}^+$ and define the directional Laplace transform:

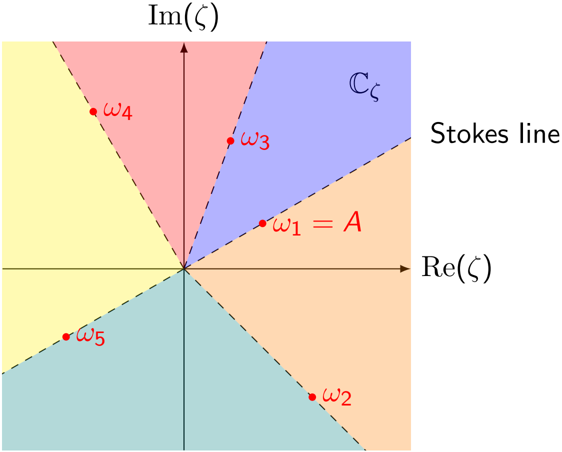
$$(\mathcal{L}_\theta \widehat{f})(z) = \int_{\gamma_\theta} e^{-z\zeta} \widehat{f}(\zeta)\, d\zeta = \int_0^{e^{i\theta}\infty} e^{-z\zeta} \widehat{f}(\zeta)\, d\zeta,$$

provided that $\widehat{f}$ can be analytically continued along γ_θ ($\approx$ resurgent)



Borel-Laplace resummation: sectorial sums, Stokes lines

Singular directions, that is Stokes lines, split $\mathbb{C}_\zeta$ in sectors $\Gamma_i = \cup_\theta \gamma_\theta$. They give rise to Stokes phenomenon in $\mathbb{C}_x$.



Borel-Laplace resummation

If all works out we have a resummation scheme $\mathcal{S}_\theta$:

$$\begin{tikzcd}
\widetilde{f}(x) \arrow[rr, "\mathcal{S}_\theta = \mathcal{L}_\theta \circ \mathcal{B}"] \arrow[dr, "\mathcal{B}"'] & & f_\theta(z) \\
& \widehat{f}(\zeta) \arrow[ur, "\mathcal{L}_\theta"'] &
\end{tikzcd}$$

$$z = \frac{1}{x}$$

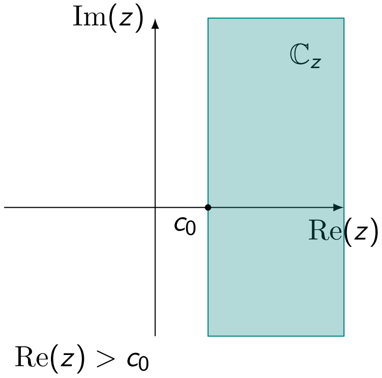
$\mathcal{S}_0$: (conventional) Borel resummation, $\gamma_0 = \mathbb{R}^+$.

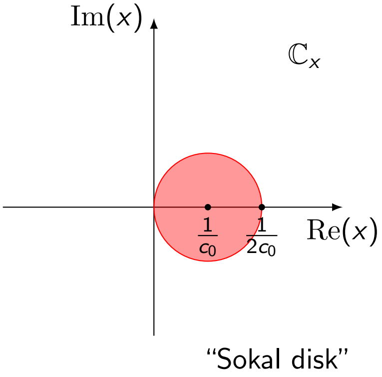
$\mathcal{B}$ and $\mathcal{L}_\theta$ are morphisms of differential algebra, $*$ product in $\mathbb{C}_\zeta$

The perturbative expansion of $f_\theta(x) = (\mathcal{S}_\theta \widetilde{f})(x)$ is $\widetilde{f}(x)$.

Borel-Laplace resummation: directionality

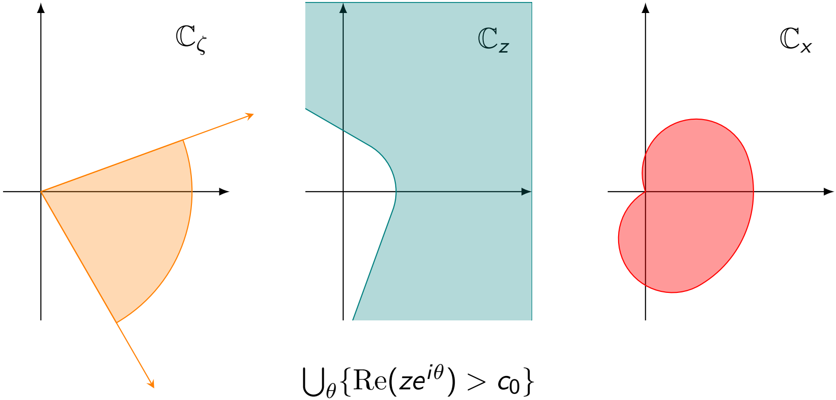
In the 0-direction $\gamma_0 = \mathbb{R}^+$.





Borel-Laplace resummation: directionality

In sectors Γ of $\mathbb{C}_\zeta$ to bean-shaped domains of $\mathbb{C}_x$.

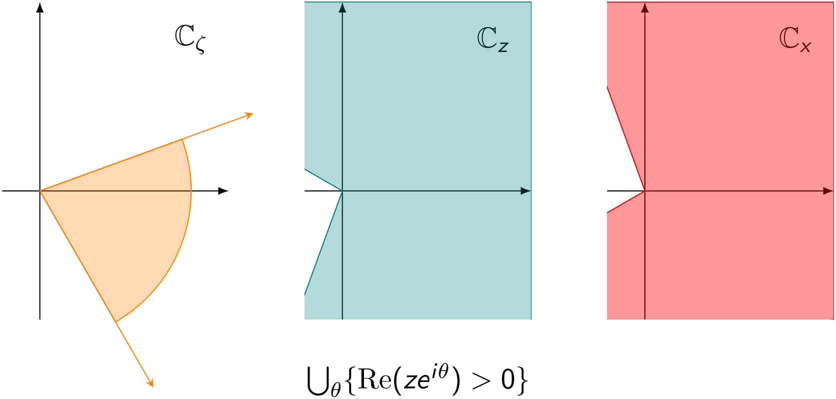


$\bigcup_\theta\{\mathrm{Re}(ze^{i\theta}) > c_0\}$

The resummation doesn't apply in all directions of $\mathbb{C}_x$ ($\neq$ convergence).

Borel-Laplace resummation: directionality

In sectors Γ of $\mathbb{C}_\zeta$ with $c_0 = 0$ to union of half-planes in $\mathbb{C}_x$.



$\bigcup_\theta \{\mathrm{Re}(ze^{i\theta}) > 0\}$

The resummation doesn't apply in all directions of $\mathbb{C}_x$ ($\neq$ convergence).

Borel-Laplace resummation: lack of unicity

If $\widetilde{f}$ is Borel resummable to f is that f "the one"?

$\Longrightarrow$ No, since different functions have the same $\widetilde{f}$.

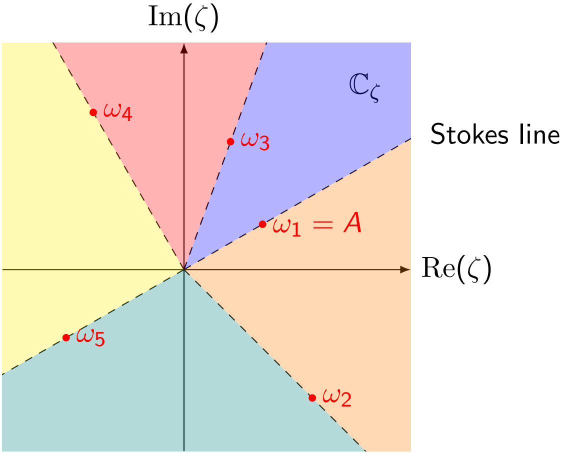
Unless stringent and *hard to show* conditions are fulfilled (Nevanlinna–Sokal theorem) there is no certainty the proper function is reproduced.

Not a problem! We already knew it was up to non-perturbative terms.

A. D. Sokal. "An improvement of Watson's theorem on Borel summability". In: *J. Math. Phys.* 21 (1980), pp. 261–263.

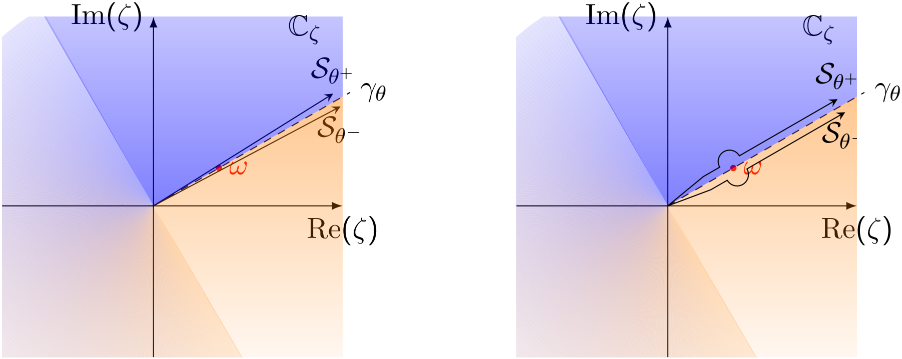
What about those Stokes lines?

Different summations due to Stokes lines. What happens there?



What about those Stokes lines?: Lateral summations

Define the lateral sums: $\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta+\epsilon}$ and $\mathcal{S}_{\theta^-} = \mathcal{S}_{\theta-\epsilon}$ with $\epsilon \to 0^+$. That is summation slightly above and below γ_θ.



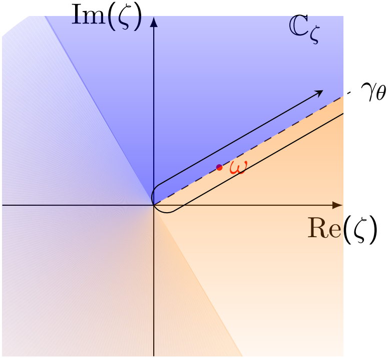
What about those Stokes lines?: discontinuity of f

$\mathcal{S}_{\theta^+}\widetilde{f}$ and $\mathcal{S}_{\theta^-}\widetilde{f}$ are different summations originating from the *same perturbative expansion* $\widetilde{f}$.

Define the discontinuity as

$$\mathrm{disc}_\theta(f)(x) = (\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-})\widetilde{f},$$

whose perturbative expansion is necessarily zero: a non-perturbative term.



The singular lines carry the non-perturbative data.

What about those Stokes lines? Euler's case

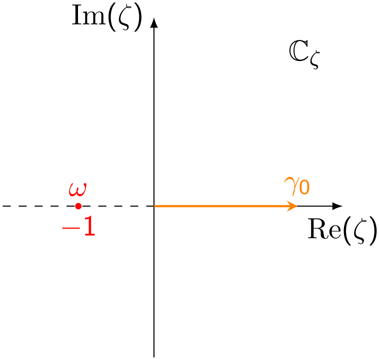
The ODE $x^2 f'(x) + f(x) = x$ has the perturbative solution

$$\widetilde{f}(x) = \sum_{n \geqslant 0} (-1)^n n! x^{n+1}$$

whose Borel transform is

$$\mathcal{B}[\widetilde{f}](\zeta) = \sum_{n \geqslant 0} (-1)^n \zeta^n = \frac{1}{1+\zeta}$$

which has a simple pole at $\zeta = -1$ and is Borel resummable.



$\gamma_0 = \mathbb{R}^+$ is non-singular,
$\gamma_\pi = \mathbb{R}^-$ is a Stokes line.

What about those Stokes lines? Euler's case

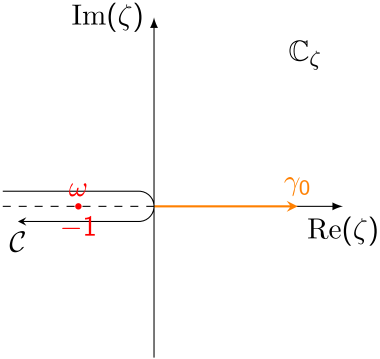
The summation on $\mathbb{R}^+$ yields

$$(\mathcal{S}_0\widetilde{f})(x) = \int_0^\infty \frac{e^{-z\zeta}}{1+\zeta}d\zeta \sim \mathrm{Ei}$$

Its discontinuity is:

$$\begin{aligned}\mathrm{disc}_\pi(f)(x) &= (\mathcal{S}_{\pi^+} - \mathcal{S}_{\pi^-})\widetilde{f} \\ &= \int_{\mathcal{C}} \frac{e^{-z\zeta}}{1+\zeta}d\zeta \\ &= -2\pi i \; e^{\frac{1}{x}},\end{aligned}$$

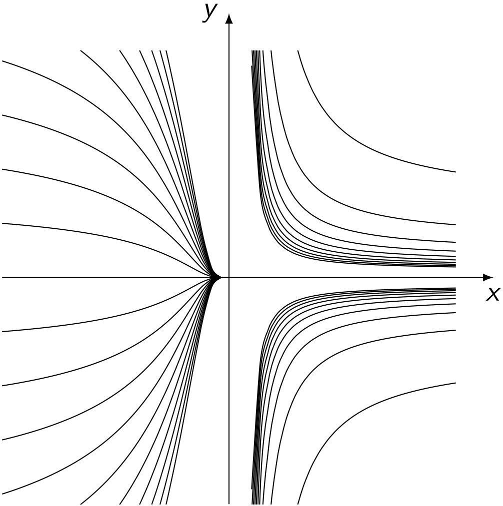
which is a solution to the homogeneous problem with a null Taylor expansion for $x \to 0$.



$e^{-\frac{\omega}{x}}$ is the prototype of a *non-perturbative* contribution.

What about those Stokes lines? Euler's case

Dramatically different behavior of $e^{\frac{1}{x}}$ with a node behavior for $x \leqslant 0$ and a saddle behavior for $x > 0$.



Experimental applied mathematics

These kind of problems can be approached ‘experimentally’ from the numerically computed sequence

$$a_0, a_1, a_2, a_3, \dots, a_{100}, \dots, a_{200}, \dots \quad \text{(the more the better [bottleneck]).}$$

The constants K, A, μ, c_1, c_2, ... appearing in the large order behavior

$$a_n \sim KA^{-(n+\mu)}\Gamma(n+\mu)\left(1+\frac{c_1}{n}+\frac{c_2}{n^2}+\dots\right), \quad n\to\infty,$$

can then be *chased* numerically using accelerations technics (Richardson transform).

This way you can conjecture their ‘true’ values from the experimental ones.

T. Sulejmanpasic and M. Ünsal. “Aspects of perturbation theory in quantum mechanics: The BenderWu Mathematica package”. In: *Comput. Phys. Commun.* 228 (2018), pp. 273–289. arXiv: `1608.08256 [hep-th]`.

Experimental applied mathematics

For the Laplace transform and from the numerically computed sequence

$a_0, a_1, a_2, a_3, \ldots, a_{100}, \ldots, a_{200}, \ldots$ (the more the better [bottleneck])

one can substitute to the truncated $\widehat{f}(\zeta)$ its Padé approximant:

$$\mathcal{P}_N(\zeta) = \left[\lfloor \frac{N}{2} \rfloor / \lfloor \frac{N+1}{2} \rfloor \right]_{\widehat{f}}(\zeta) = \frac{\sum_{i=0}^{\lfloor \frac{N}{2} \rfloor} \alpha_i \zeta^i}{1 + \sum_{j=1}^{\lfloor \frac{N+1}{2} \rfloor} \beta_j \zeta^j}$$

thus locating poles in $\mathbb{C}_\zeta$ and log-cuts (from the 'condensation' of poles).

Rmk. Padé approximants in the Borel plane (then back through Laplace) $\equiv$ Borel-Padé (superior to Padé in the x plane)

A “simple” example: the anharmonic oscillator (AO)

Consider the anharmonic oscillator (AO):

$$\widehat{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2 + gx^4,$$

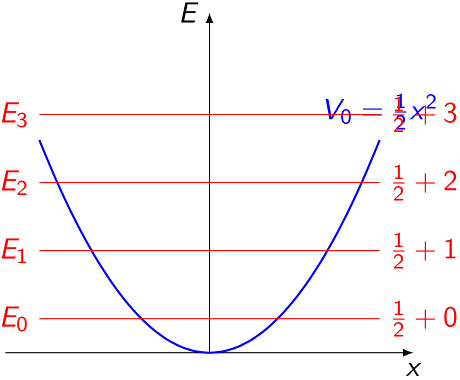
with $\hbar = 1$, $m = 1$, $\omega = 1$ and the eigenvalue problem

$$\widehat{H}\psi(x) = E(g)\psi(x)$$

with $\psi(x)$ fulfilling the boundary condition

$$\lim_{|x|\to\infty} \psi(x) = 0.$$

AO: $g = 0$ reduction to the HO



For $g = 0$ one has the usual harmonic oscillator with:

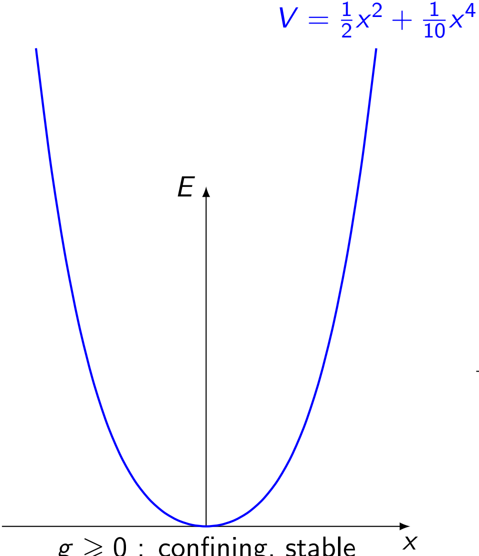
$$\psi_n(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x),$$

$$E_n = n + \frac{1}{2},$$

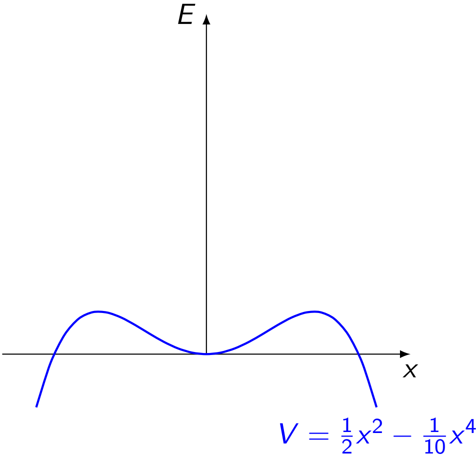
with in peculiar the ground state:

$$E_{gr}(g = 0) = \frac{1}{2}.$$

AO: Two dramatically different cases



$g \geqslant 0$: confining, stable



$g < 0$: unstable

AO: Bender and Wu 'tour de force'

Through *very efficient methods* Bender and Wu computed the first 75 coefficients of $\widetilde{E}_{gr}(g)$:

$$\widetilde{E}_{gr}(g) = \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4 + \ldots$$

and determined its large order behavior:

$$a_n \sim -\frac{\sqrt{6}}{\pi^{3/2}}(-3)^n \Gamma(n+1/2), \quad n \to \infty$$

C. M. Bender and T. T. Wu. "Anharmonic Oscillator". In: *Phys. Rev.* 184 (1969), 1231–1260, J. J. Loeffel et al. "Pade approximants and the anharmonic oscillator". In: *Phys. Lett. B* 30.9 (1969), 656–658, C. M. Bender and T. Tsun Wu. "Large-Order Behavior of Perturbation Theory". In: *Phys. Rev. Lett.* 27 (1971), 461–465.

AO: Bender and Wu 'tour de force'

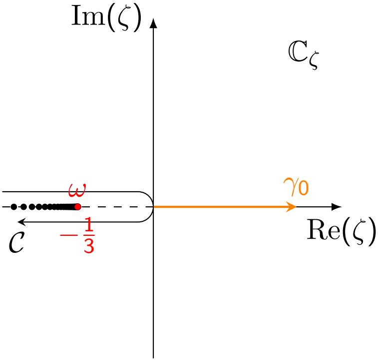
For $N = 53$ the summation on $\mathbb{R}^+$ yields

$$(\mathcal{L}_0 \circ \mathcal{P}_N \circ \mathcal{B}\widetilde{E}_{gr})(g = 1) = 0.8037706512,$$

exact up to 10 decimal digits.

The singularities accumulate from $-\infty$ up to $\omega = A = -\frac{1}{3}$ leading to a non-perturbative term:

$$E_{gr}^{np} \approx (-g)^{-\frac{1}{2}} e^{\frac{1}{3g}}.$$



J. J. Loeffel et al. "Pade approximants and the anharmonic oscillator". In: *Phys. Lett. B* 30.9 (1969), 656–658, S. Graffi, V. Grecchi, and B. Simon. "Borel summability: Application to the anharmonic oscillator". In: *Physics Letters B* 32.7 (1970), 631–634, E. Caliceti et al. "From useful algorithms for slowly convergent series to physical predictions based on divergent perturbative expansions". In: *Phys. Rept.* 446 (2007), pp. 1–96. arXiv: `0707.1596 [physics.comp-ph]`.

Extending perturbation theory: transseries

For a reconstruction of f one can take the resummed perturbative expansion and its non-perturbative completion building the transseries:

$$f(x) = f^{(0)}(x) + \sum \sigma_i e^{-\frac{\omega_i}{x}} f^{(\omega_i)}(x),$$

σ_i are constants, $f^{(0)}$ (the perturbative term) and the $f^{(\omega_i)}$ (also) have factorially divergent perturbative expansion:

$$\widetilde{f}^{(\omega_i)}(x) = \sum_{n \geqslant 0} a_n^{(i)} x^n.$$

and are given by the behavior or $\widehat{f}$ near ω_i in $\mathbb{C}_\zeta$.

This can be taken as an *ansatz* and brute-forced into your problem.

⚠ The $e^{-\omega_i/x}$ which are vanishingly small for $x \in \mathbb{R}^+$ can become dominant in $\mathbb{C}_x$, in particular in crossing from $x > 0$ to $x < 0$.

Large order behavior of the perturbative term

From $\widetilde{f}$ we can locate poles and cuts in $\mathbb{C}_\zeta$ and thus the $e^{-\omega/x}$.

It goes further than this!

Assuming the first singular point in $\mathbb{C}_\zeta$ is $\omega_1 = A$ and inspecting the large order behavior of the a_n gives:

$$\begin{aligned} a_n &\sim \frac{S}{2\pi} \sum_k A^{k-n} a_k^{(1)} \Gamma(n-k) \\ &\sim \frac{S}{2\pi} A^{-n} \Gamma(n) \left(a_0^{(1)} + \frac{A}{n-1} a_1^{(1)} + \frac{A^2}{(n-1)(n-2)} a_2^{(2)} + \dots \right) \end{aligned}$$

the other function $f^{(1)}$ (re)appears in the late terms of $f^{(0)}$: it 'resurges'!

That is, *which is reborn or manifests itself again after a period of disappearance or absence.*

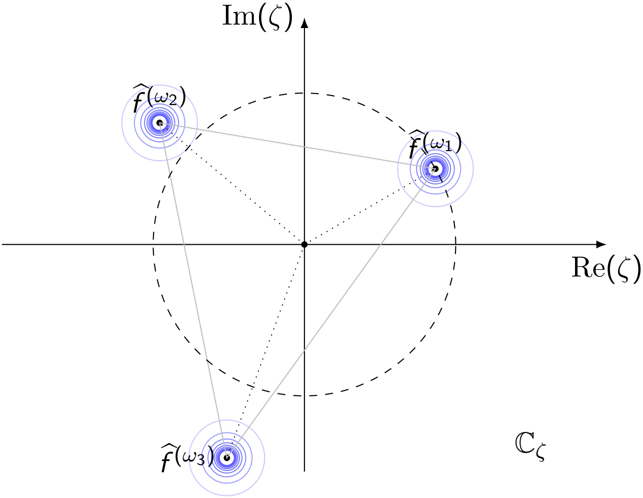
The resurgence property

(...) en ses singularités le mineur $\varphi(\zeta)$ a un comportement qui reproduit ou qui rappelle celui de $\varphi(\zeta)$ à l'origine. D'une façon imagée, on peut dire que la fonction originelle "ressuscite" en ses singularités – telle quelle ou plus ou moins altérée (...). D'où le nom de fonctions résurgentes donné à ces fonctions (...).

(...) in its singularities the minor $\varphi(\zeta)$ has a behavior that reproduces or recalls that of $\varphi(\zeta)$ at the origin. In a figurative way, we can say that the original function "resurrects" in its singularities – as is or more or less altered (...). Hence the name resurgent functions given to these functions (...).

J. Écalle. "Les fonctions résurgentes". In: *Publ. Math. Orsay* Vol. I-II-III (1981).

The resurgence property



Resurgent functions and Alien Calculus

This is the manifestation of the *late term/early term relationship* between different sectors.

Resurgence's credo

Physical observables are resurgent transseries whose sectors are connected among themselves in an intricate way which can be disentangled from the simplest of them $f^{(0)}$. The technical tool to do so is Alien Calculus.

In one case an even better *early term/early term* correspondence has been established.

G. V. Dunne and M. Unsal. "Generating nonperturbative physics from perturbation theory". In: *Phys. Rev. D* 89.4 (2014), p. 041701. arXiv: `1306.4405 [hep-th]`, G. V. Dunne and M. Unsal. "Uniform WKB, Multi-instantons, and Resurgent Trans-Series". In: *Phys. Rev. D* 89.10 (2014), p. 105009. arXiv: `1401.5202 [hep-th]`.

Conclusion

1. There is non-perturbative data encoded in perturbative series.
   $\longrightarrow$The K, A, μ and everything from the large order.
2. Resurgence (theory and Alien Calculus) can decipher this data and enhance perturbation theory.

Perturbation theory is a way to reach non-perturbative theory as they are linked.

What's next?

- Alien calculus
- Uniform-WKB going to Exact-WKB/Exact quantization
- non-perturbative ambiguity
- Hyperasymptotics to get faithful numerical approximations
- relation to Lefshetz thimbles in QFT
- ...

Thank you!

F. Fauvet, F. Menous, and J. Queva. "Resurgence and holonomy of the ϕ^{2k} model in zero dimension". In: *J. Math. Phys.* 61.9 (2020), p. 092301. arXiv: `1910.01606 [math-ph]`.