Resurgence of the $\lambda\phi^{2k}$ model in zero dim QFT

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joint work with F. Fauvet & F. Menous

Ref. – Arxiv: 1910.01606
- Rivasseau & Wang JMP 2010
- Rivasseau & Lionni Math Phys. Anal. Geom 2018

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With

$$Z_0(\lambda) = \int_{\mathbb{R}} e^{-\frac{1}{2}\phi^2 - \lambda\phi^{2k}} \frac{d\phi}{\sqrt{2\pi}}, \quad k = 2, 3, \dots$$

! no external current j

$$W(\lambda) = \log Z_0(\lambda)$$

Proof of summability of Z and W using intermediate field rep. and LVE $\Rightarrow$ painful bounds and delicate combinatorics

Aim. not do that! Retrieve these infos from the differential equations fulfilled by $Z_0(\lambda)$

Set $$Z_0 = \int_{\mathbb{R}} e^{-\frac{1}{2}\phi^2 - \lambda\phi^{2k}} \frac{d\phi}{\sqrt{2\pi}}$$

$$Z_{2j} = \int_{\mathbb{R}} \phi^{2j} e^{-\frac{1}{2}\phi^2 - \lambda\phi^{2k}} \frac{d\phi}{\sqrt{2\pi}}, \quad j = 0, 1, 2, \dots$$

$Z_{2j} \simeq$ moments, correlation functions, ...

Prop. $Z_{2j}(\lambda)$: well-behaved, continuous, analytic, unique asymptotic / perturbative behavior

$\underline{\underline{\lambda > 0}}$

Set $k = 2$ to simplify here

The ODE: – differentiating under $\int$
– integration by parts

$$\left\{\begin{array}{l} \partial_\lambda Z_{2j} = -Z_{2j+4} \\ (2j+1)\, Z_{2j} = Z_{2j+2} + 4\lambda Z_{2j+4} \end{array}\right.$$

infinite set of DSE in zero dim

H: $$\boxed{\left[(3+4\lambda\partial_\lambda)(1+4\lambda\partial_\lambda) + \partial_\lambda\right] Z_0 = 0}$$

the source of everything

The Newton polygone (NP) of E:

$\rightsquigarrow$ pictorial tool to gain info on E

$\} \; \theta = x \frac{d}{dx}$

Let $H(f) = \left(H_n(x)\, \theta^n + \cdots + H_1(x)\,\theta + H_0(x) \right) f$

$\} \; f$ holonomic

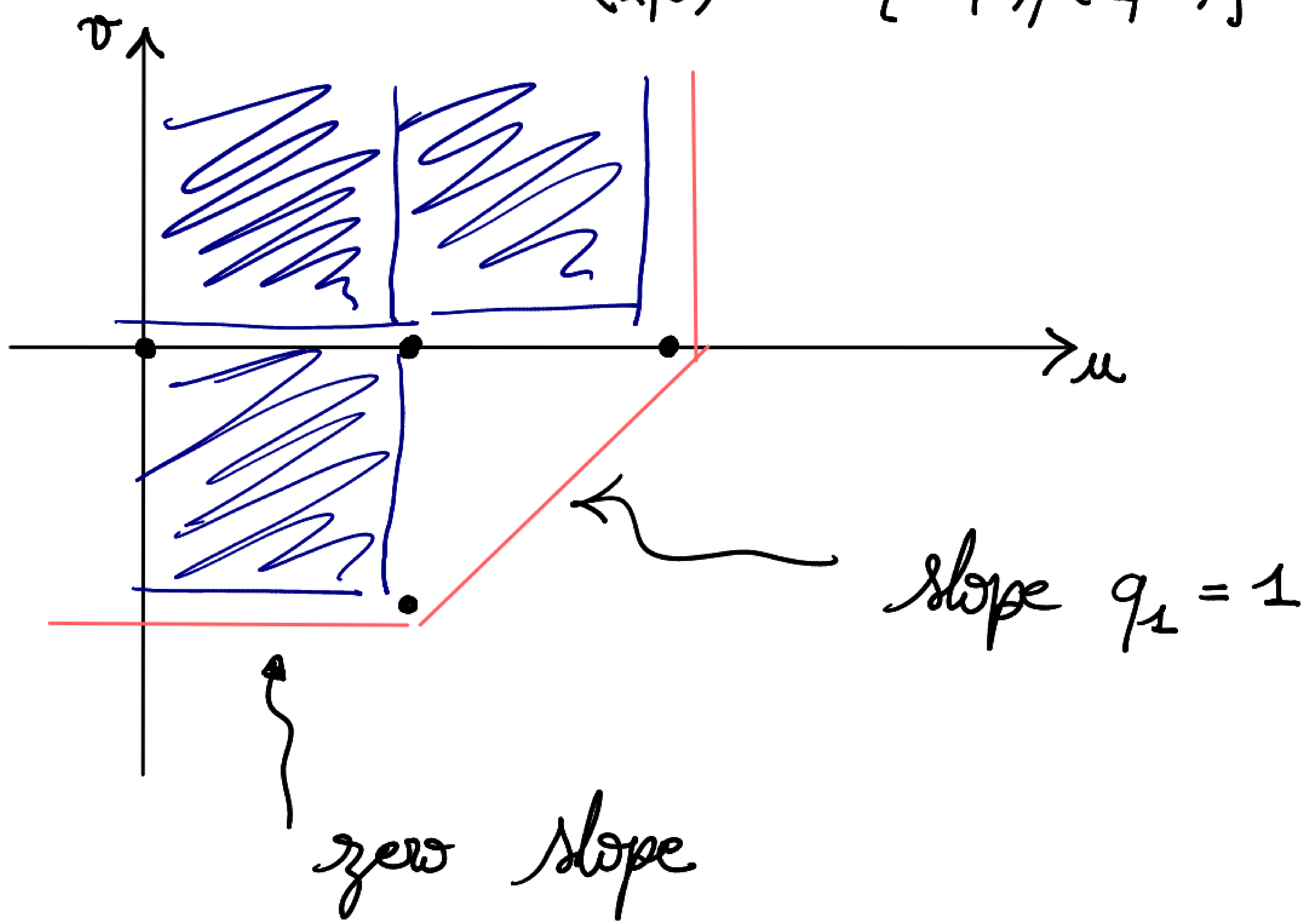
$\mathcal{S} = \{ (i, q) \in \{0, \dots, n\} \times \mathbb{Q}$, the monomial x^q appears in $H_i(x) \}$

$M^+_{(i,q)} = \{ (u, v) \in \mathbb{R}^+ \times \mathbb{R}, \; 0 \leq u \leq i, \; q \leq v \}$

$\hookrightarrow$ NP of H at zero = Convex hull of $\left(\bigcup_{(i,q) \in \mathcal{S}} M^+_{(i,q)} \right)$

setting $x = \lambda$: $H = (3+4\theta_x)(1+4\theta_x) + x^{-1}\theta_x$

$$= \underbrace{16\,\theta_x^2}_{(2,0)} + \underbrace{(16 + x^{-1})\theta_x}_{\{(1,0),\,(1,-1)\}} + \underbrace{3}_{(0,0)}$$



Main theorem for Gevrey asymptotics

- $H(f)$ has formal sol $\tilde{f}$ iff its NP at 0 has an horizontal slope
- for $q_1 < q_2 < \dots < q_r$ positive slopes near an irregular-singular point the determining factors are $e^{\alpha / x^{q_i}}$
- gives the maximal growth near the origin

$$q_0 = 0 < q_1 = 1$$

formal sol $\qquad e^{\alpha/\lambda} \qquad$ ("1-instanton")

<u>Algo to "study" H</u>

① Get the formal sol $\tilde{Z}$ using Frob. method

② Change the diff. op. $H \to H_u$, $u \in \mathbb{C}$, with $H_u = e^{-\frac{u}{x}} H e^{\frac{u}{x}}$
and u s.t. $P(u) = 0$, $P(u) \equiv$ lowest degree in x in H_0
$\equiv$ pol. of order n

$\boxed{\text{⊕ indicial eq. } x^{-\beta} H e^{\beta}}$

③ If $P(u)$ has n distincts roots $NP(H_u)$ has the same shape as $NP(H)$

$$\Rightarrow \left\{ \tilde{f}_0(x),\ e^{u_1/x^{q_1}} \tilde{f}_1(x), \dots, e^{u_{n-1}/x^{q_{n-1}}} \tilde{f}_{n-1}(x) \right\} \text{ a } \underline{\text{basis}} \text{ of sol.}$$

Example. $NP(H)$ has a $q_1 = 1$ slope

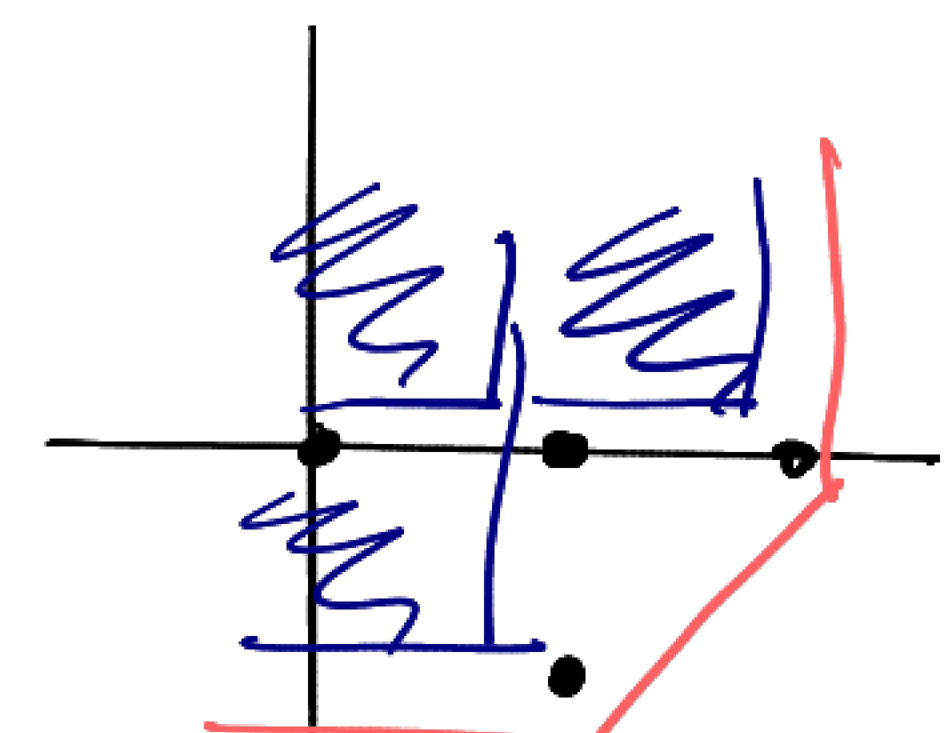
Set $H_u = e^{-u/x} H e^{u/x}, \quad u \in \mathbb{C}$

$$= e^{-u/x} \left\{ 16\,\theta_x^2 + (16 + x^{-1})\theta_x + 3 \right\} e^{u/x}$$

$$= 16\,\theta_x^2 + \left\{ 16 + (1-32u)\,x^{-1} \right\} \theta_x + \left\{ 3 + u(16u-1)\,x^{-2} \right\} \times 1$$

$\Rightarrow P(u) = u(16u-1)$

$\Leftrightarrow$ non-trivial root $u_1 = \frac{+1}{16}$



$\Rightarrow H_{\frac{1}{16}} = 16\,\theta_x^2 + (16 - x^{-1})\theta_x + 3$

horizontal slope

$e^{\frac{1}{16x}} \tilde{Z}^{(1)}$ non-perturbative sol

Resurgence for holonomic function with 1-critical time

Def. a function, or a formal series, is holonomic if it is a sol of a linear ODE $H(f)=0$ $\Big\{$ $H(\tilde{Z}_0^{(0)})=0$ $\Big\{$ $H_{116}(\tilde{Z}_0^{(1)})$

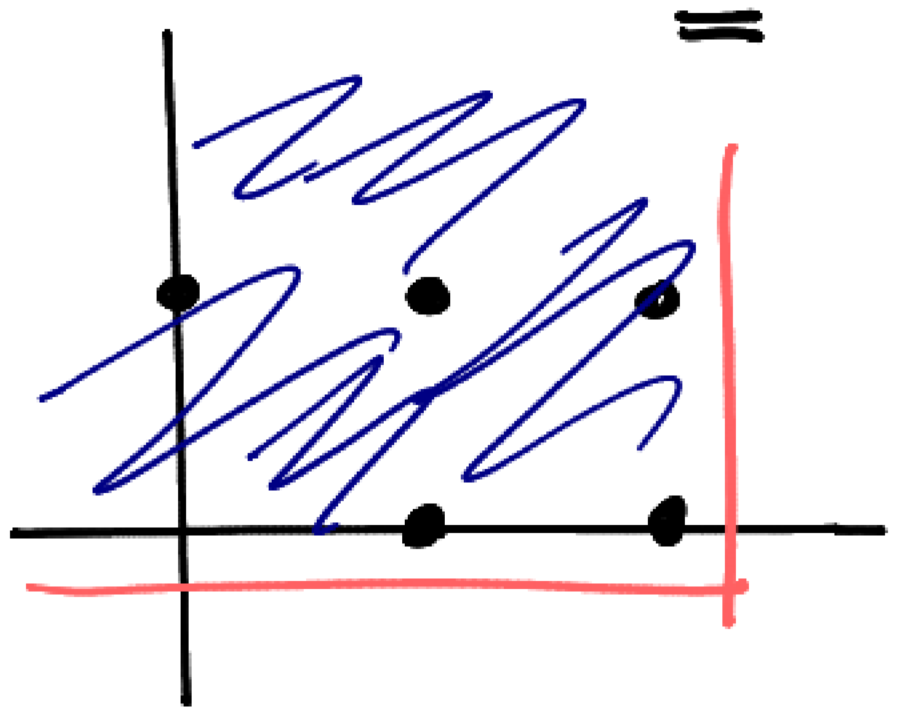
Prop.
- sum and product of holonomic functions are holonomic
- post composition of an holonomic function by an algebraic function is holonomic $\Big\{$ $g(x)=f(x^r),\ r\in\mathbb{Q}$

Prop. The Borel transform of an holonomic function $f\in x\mathbb{C}[[x^{\mathbb{Q}^+}]]$ with vanishing constant is holonomic $\Rightarrow$ It is resurgent with finite number of singularities

Here turn H from the x-plane to the Borel ξ-plane

$$H \rightsquigarrow \hat{H} = (16\xi+1)\,\xi\,\partial_\xi^2 + 2(32\xi+1)\,\partial_\xi + 35\xi$$

$$= (16\xi+1)\,\theta_\xi^2 + (48\xi+1)\,\theta_\xi + 35\xi$$



$\rightsquigarrow$ singularities at $\omega = \frac{-1}{16} \to \dot{\Delta}_{-1/16}$

NP at zero, flat $\Rightarrow$ analytic solutions at 0 that can be continued along any path avoid $\omega = \frac{-1}{16}$

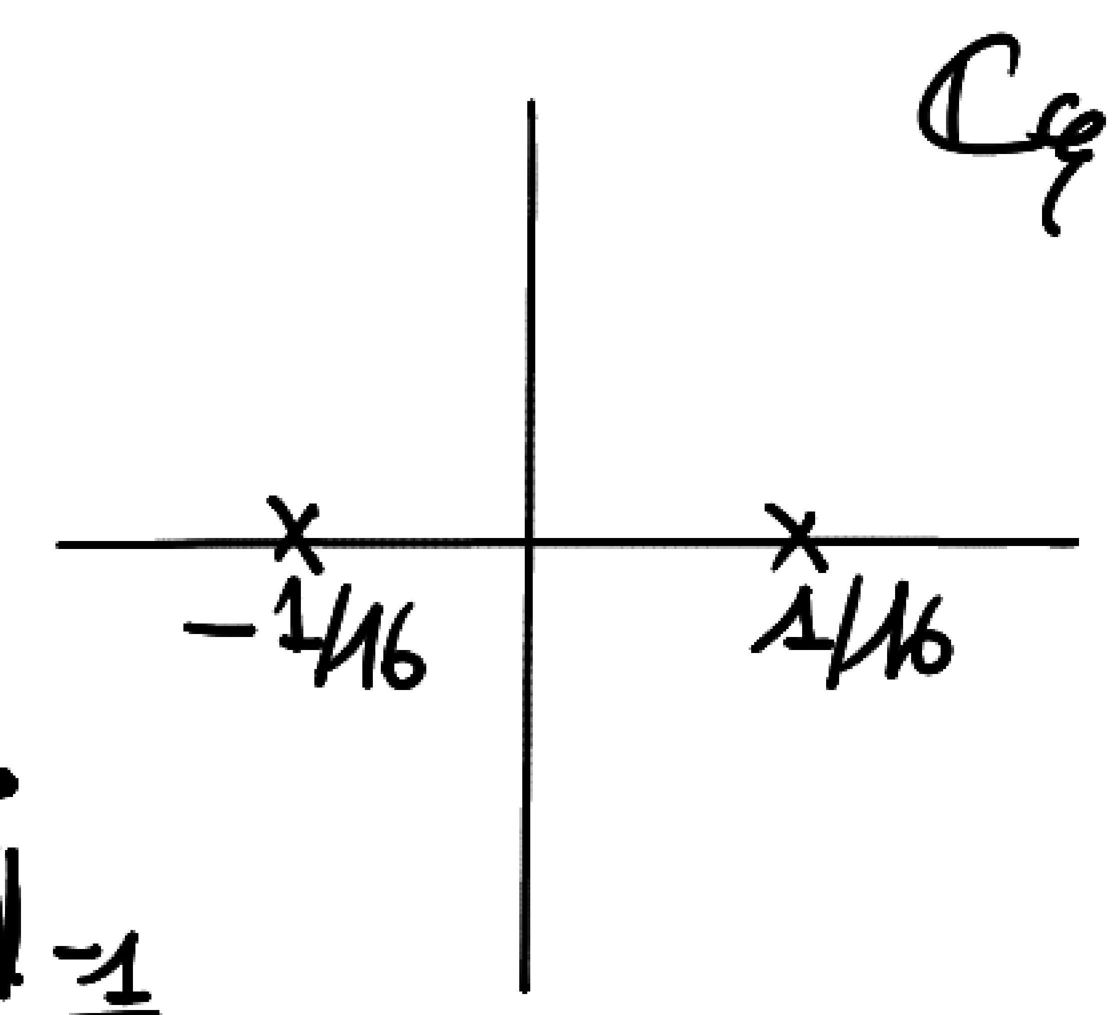
NP at ∞ $\zeta = 1/\xi$: no negative slope $\Rightarrow$ sols are $O(|\xi|^{-\beta})$

$\Rightarrow$ The LP of $\hat{Z}^{(0)}$ exists in any dir avoiding $\omega = -1/16$

Same for $H_{1/16} \rightsquigarrow \hat{H}_{1/16}$ with singularity at $\omega = \frac{+1}{16} \rightsquigarrow \dot{\Delta}_{1/16}$

Finally :

$$Z_0(x) = \sigma_0 \tilde{Z}_0^{(0)}(x) + \sigma_1 e^{\frac{1}{16x}} \tilde{Z}_0^{(1)}(x)$$



$\sigma_0, \sigma_1 \in \mathbb{C}$

on which only two alien derivatives $\dot{\Delta}_{\frac{1}{16}}$, $\dot{\Delta}_{\frac{-1}{16}}$

$$\boxed{\dot{\Delta}_\omega Z_0(x) = \left(A_0 \sigma_0 \frac{\partial}{\partial \sigma_0} + A_1 \sigma_1 \frac{\partial}{\partial \sigma_1} \right) Z_0(x)}$$

instance of the bridge equation

Nonlinear operations

Prop. Let φ be a resurgent function with $\tilde{\varphi}(z) \in z^{-1}\mathbb{C}[[z^{-1}]]$ and χ an analytic function, then : ① $\chi \circ \varphi$ is resurgent

② $\forall \omega \in \mathbb{C}^* : \Delta_\omega(\chi \circ \varphi) = (\partial_z \chi) \circ \varphi \; \Delta_\omega \varphi$

Example $\tilde{Z}_0(\lambda) - 1 \in \lambda\,\mathbb{C}[[\lambda]] \Rightarrow W(\lambda) = \log(\tilde{Z}_0(\lambda))$ is resurgent

$\rightsquigarrow$ There is an infinite set of singularities for W : $\Omega = \frac{\mathbb{Z}}{16}$ and two acting alien derivatives : $\Delta_{7/16}, \Delta_{1/16}$

(with composition)