A coherent state approach to the quantization of the particle in the infinite well and of the fuzzy hyperboloid

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A reminder on coherent states (1)

Consider the harmonic oscillator:

$$\mathcal{H} = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 Q^2.$$

In its Heisenberg rep. the position op. reads:

$$Q(t) = e^{\frac{i}{\hbar}\mathcal{H}t} Q e^{-\frac{i}{\hbar}\mathcal{H}t}$$

then one can find states $|z\rangle$, $z \in \mathbb{C}$, s.t. in the mean one recovers the classical motion

$$\langle z|Q(t)|z\rangle = \sqrt{\frac{2\hbar}{m\omega}}\, |z| \cos(\omega t + \varphi)$$

Such states are given through the decomposition:

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} \phi_k \qquad (z \in \mathbb{C},\ \phi_k \simeq \text{Hermite pol.})$$

A reminder on coherent states (2)

Then, those states fulfill 4 remarkable properties

- they saturate Heisenberg's inequality

$$\langle \Delta Q \rangle_z \langle \Delta P \rangle_z = \frac{\hbar}{2}$$

where $\langle Q \rangle_z = \sqrt{\langle z|Q^2|z\rangle - \langle z|Q|z\rangle^2}$,

- they are eigenstates of an annihilation operator

$$a|z\rangle = z|z\rangle, \quad z \in \mathbb{C}$$

with $a = (2m\hbar\omega)^{-\frac{1}{2}}(m\omega Q + iP)$.

A reminder on coherent states (2)

- they belong to the orbit of a "vacuum" $|0\rangle$ through the unitary action of the Weyl-Heisenberg group
$$|z\rangle = e^{za^\dagger - \bar{z}a}|0\rangle.$$

- they form an overcomplete basis in the Hilbert space of the harmonic oscillator, this is shown by the identity resolution:
$$\mathbb{I} = \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z| dxdy,$$
with $z = x + iy$, $x, y \in \mathbb{R}$.

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$\longrightarrow$ starting point of the generalized Coherent States

Generalized coherent states

(scalar case 1)

Consider a space X along with its measure $\mu(\mathrm{d}x)$ and $L^2(X,\mu)$ the Hilbert space of the (eq. class of) functions which are square integrable $f(x)$ on X :

$$|f|^2 = \int_X |f(x)|^2 \, \mu(\mathrm{d}x) < \infty$$
$$\langle f_1 | f_2 \rangle = \int_X \overline{f_1(x)} f_2(x) \, \mu(\mathrm{d}x) \, .$$

The simplest being to think of X as a phase space.

Ali, S.T. and Antoine, J.-P. and Gazeau, J.-P., *Coherent States, Wavelets and Their Generalizations*, Springer-Verlag, (2000).

J.-P. Gazeau, Coherent states in quantum physics, Wiley-VCH, (2009).

Generalized coherent states

(scalar case 2)

Out of $L^2(X, \mu(\mathrm{d}x))$ one extract an ONB $S = \{\phi_n(x)\}_n$ which spans an Hilbert space $\mathscr{H}_S$ and fulfills the crucial (!) property:

$$0 < \mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \quad \text{(ae)}$$

Then one construct the (scalar) coherent state $\{|x\rangle\}_{x\in X}$ in $\mathscr{H}_S$ via the linear combination:

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \overline{\phi_n(x)} |n\rangle \,,$$

Generalized coherent states

(scalar case 3)

Those states then automatically verify:

- normalization property:

$$\langle x|x\rangle = 1$$

- identity resolution in $\mathscr{H}_S$:

$$\int_X |x\rangle\langle x|\,\mathcal{N}(x)\,\mu(\mathrm{d}x) = \mathbb{I}_{\mathscr{H}_S}$$

Comments

- States $|x\rangle$ form in general an “overcomplete basis” in $\mathscr{H}_S$ (total but not free).
- To any vector $|\phi\rangle$ in $\mathscr{H}_S$ is isometrically associated the function in $L^2(X, \mu(dx))$:

$$\phi(x) \equiv \sqrt{\mathcal{N}(x)}\langle x\,|\phi\rangle$$

- This function obeys

$$\phi(x) = \int\limits_X \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x|x'\rangle\phi(x')\,\mu(dx')$$

- So $\mathscr{H}_S$ is isometric to a reproducing Hilbert space with kernel

$$\mathcal{K}(x,x') = \sqrt{\mathcal{N}(x)\mathcal{N}(x')}\langle x\,|x'\rangle.$$

Generalized coherent states

(scalar case 4)

Then, with those (scalar) coherent states one can quantize the classical observables to their quantum counterpart (⇔ map functions into endomorphisms of $\mathscr{H}_S$)

$$\widehat{f} := \int_X f(x)|x\rangle\langle x|\,\mathcal{N}(x)\,\mu(\mathrm{d}x)$$

the resulting operator $\hat{f}$ is customarily called the *upper symbol* of f. In the above it is blatant that the *quantizability* condition amounts to an integrability condition of the classical observable.

Then, one can compute the mean value of such operators in CS (*lower symbol*):

$$\check{f}(q,p) = \langle x|\hat{f}|x\rangle.$$

(generally different from the classical function f we started with)

Generalized coherent states

(vectorial/tensorial case 1)

This scheme can be further generalized. Instead of using simply $\mathscr{H}_S \subset L^2(X, \mu(\mathrm{d}x))$ consider $\mathfrak{K} \otimes \mathscr{H}_S$
Let

- ϕ_k be a ONB of $\mathscr{H}_S$,
- χ_i be a ONB of $\mathfrak{K}$, with $\mathfrak{K}$ separable and finite.

Let us denote by $\mathfrak{B}_2(\mathfrak{K})$ the vector space of of Hilbert-Schmidt operators.

S.T. Ali, M. Englis, and J.-P. Gazeau. J. Phys. A, 37, (2004). arXiv:math-ph/0311042

Generalized coherent states

(vectorial/tensorial case 2)

Then let us consider $F_k : X \to \mathfrak{B}_2(\mathfrak{K})$ a linear map which fulfills

- finiteness, $\forall x \in X$:

$$0 < \mathcal{N}(x) = \sum_{k=0}^{\dim \mathfrak{K}} |F_k(x)| < +\infty \qquad (ae),$$

  where $|F_k(x)|^2 = |F_k(x)F_k^\dagger(x)|$ is the positive part of the operator $F_k(x)$,
- identity resolution in $\mathfrak{K}$:

$$\int_X F_k(x)F_l^\dagger(x)\mu(\mathrm{d}x) = \delta_{kl}\mathbb{I}_{\mathfrak{K}}$$

Generalized coherent states

(vectorial/tensorial case 3)

Then one constructs coherent states:

$$|x, i\rangle = \frac{1}{\sqrt{\mathcal{N}}(x)} \sum_{k=0}^{\dim \mathfrak{K}} F_k(x)\chi^i \otimes \phi_k$$

which then fulfills:

- normalization: $\sum_i \| \, |x, i\rangle \|^2 = 1$,
- identity resolution:

$$\sum_{i=1}^{\dim \mathfrak{K}} \int_X \mathcal{N}(x)|x, i\rangle\langle x, i|\mu(\mathrm{d}x) = \mathbb{I}_{\mathscr{H}_S} \otimes \mathbb{I}_{\mathfrak{K}}$$

Generalized coherent states

(vectorial/tensorial case 4)

Then one quantize the classical observables in a fashion similar to the previous case:

- *upper symbol*:

$$\hat{f} = A_f = \sum_{i=1}^{\dim \mathfrak{K}} \int_X f(x) |x, i\rangle\langle x, i| \mathcal{N}(x) \mu(\mathrm{d}x)$$

- *lower symbol*:

$$\check{f} = \check{A}_f = \sum_{i=1}^{\dim \mathfrak{K}} \langle x, i|\hat{f}|x, i\rangle$$

The particle in the infinite well

First canonical steps

One considers a particle trapped in an infinite potential, the Hilbert space of interest is:

$$\mathscr{H} = \{\Psi \in L^2([0,L], \mathrm{d}x) \quad | \quad \|\Psi\| = 1\}.$$

In addition, one has to make sense of *boundary conditions* $\Psi(0) = \Psi(L) \sim 0$.
In the canonical quantization scheme one considers the following realization of the canonical variables:

$$\hat{x}\Psi(x) = x\Psi(x), \qquad\qquad \hat{p}\Psi(x) = -\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}x}\Psi(x).$$

which *formally* fulfill: $[\hat{x}, \hat{p}]\Psi = \mathrm{i}\hbar\Psi$. Finally one might check the symmetry of $\hat{p}$:

$$\langle \Psi, \hat{p}\Psi' \rangle = \langle \hat{p}\Psi, \Psi' \rangle \qquad \Psi, \Psi' \in \mathscr{H}$$

w.r.t. to the usual L^2 product.

The particle in the infinite well

First canonical steps

Then one pays attention to the Hamiltonian of the free particle:

$$\mathcal{H} = -\frac{\hbar}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$$

for which a set of eigenfunctions is easily found:

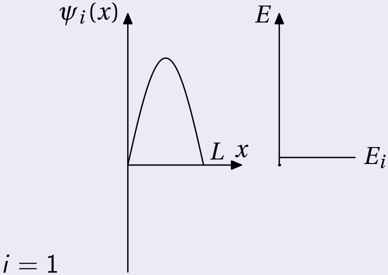
$$\Psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{\pi n}{L}x\right), \quad n \in \mathbb{N}^*$$

s.t.

$$\mathcal{H}\Psi_n = E_n\Psi_n, \qquad E_n = \frac{\hbar}{2m}\left(\frac{\pi n}{L}\right)^2.$$

The particle in the infinite well

First canonical steps



The particle in the infinite well

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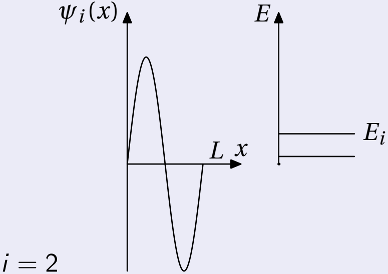

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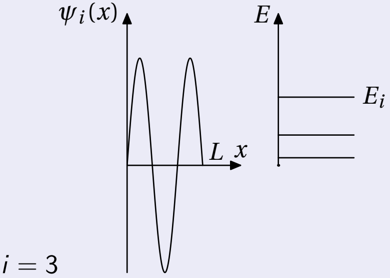

The particle in the infinite well

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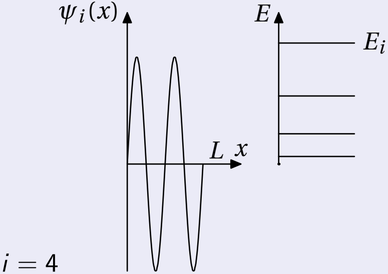

The particle in the infinite well

Troubles develop

However, weird properties arise, say, for:

$$\Phi(x) = \sqrt{\frac{30}{L^5}}x(x-L) = -\sqrt{30}\sum_{p=1}^{\infty}\left(\frac{2}{\pi(2p+1)}\right)^3 \Psi_{2p+1}(x)$$

One might be interested in its energy deviation:

$$\langle E\rangle = \langle\Phi, \mathcal{H}\Phi\rangle = 5\frac{\hbar}{mL^2}$$

$$\langle E^2\rangle = \langle\Phi, \mathcal{H}^2\Phi\rangle = 0 = \langle\mathcal{H}\Phi, \mathcal{H}\Phi\rangle = 30\left(\frac{\hbar}{mL^2}\right)^2 !!!$$

In fact, one faces problems of operator domains in Hilbert space. More precisely:

$$\mathcal{H}\Phi \notin \mathrm{dom}(\mathcal{H}) \subset \mathscr{H}$$

Bonneau, G. and Faraut, J. and Galliano, V., Am. J. Phys. **69** (3), 322 - 331, (2001).

de León, G. and Gazeau, J-P. and Gitman, D. and Queva, J., *Infinite Well: on the quantization problem*.

The particle in the infinite well

A mathematical cure

One first realizes that both the Hamiltonian and the momentum operator in spite of being symmetric are *not* self-adjoint.
To solve this problem one has to rely on vNeumann theory of self-adjoint extensions of symmetric operators.

However, there are an infinite manner to extend such operators ...

Reed, M. and Simon, B., *Methods of Modern Mathematical Physics, II. Fourier Analysis, Self-Adjointness*, Academic Press, (1975)

CS quantization of the Infinite Well

The general Hilbert space

Let us consider the phase space X of the particle:

$$X = [0, L] \times \mathbb{R} = \{x = (q, p) \mid q \in [0, L]\, , p \in \mathbb{R}\}$$

and the Hilbert space hosting the coherent states:

$$L^2_{\mathbb{C}^2}(X, \mu(\mathrm{d}x)) \simeq \mathbb{C}^2 \otimes L^2_{\mathbb{C}}(X, \mu(\mathrm{d}x)) = \left\{ \Phi(x) = \begin{pmatrix} \phi_+(x) \\ \phi_-(x) \end{pmatrix},\ \phi_\pm \in \mathrm{L}^2_{\mathbb{C}}(X, \mu(\mathrm{d}x)) \right\}.$$

García de León, P.L. and Gazeau, J.-P. and Queva, J., Phys. Lett. A **372**, 3597-3607 (2008), arXiv:0801.3769

CS quantization of the Infinite Well

Selecting the subspace

Now, let us extract a subset of vectors which will span the coherent states:

$$\Phi_{n,+}(x) = \begin{pmatrix} \phi_{n,+}(x) \\ 0 \end{pmatrix}, \qquad \Phi_{n,-}(x) = \begin{pmatrix} 0 \\ \phi_{n,-}(x) \end{pmatrix},$$
$$\phi_{n,\kappa}(x) = \sqrt{c}\, e^{-\frac{1}{2\rho^2}(p-\kappa p_n)^2} \sin\left(n\pi \frac{q}{L}\right), \qquad \kappa = \pm,\ n = \mathbb{N}^*$$

with

$$c = \frac{2}{\rho L \sqrt{\pi}}, \qquad p_n = \sqrt{2mE_n} = \frac{\hbar\pi}{L}\, n,$$

Notice the parameter ρ which has been used s.t. the property $\mathcal{N} < \infty$ is fulfilled. Interesting properties due to its presence will happen.

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CS quantization of the Infinite Well

Quantizing the classical observables

One can quantize the momentum operator:

$$\widehat{p} = \sum_{n=1}^{\infty} p_n \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes |n\rangle\langle n| \,,$$

which is self-adjoint in $\mathscr{H}_S$.

Similarly, the square of p might be quantized:

$$\widehat{p^2} = \frac{\rho^2}{2}\mathbb{I}_{\mathscr{H}_S} + \sum_{n=1}^{\infty} p_n^2 \, \mathbb{I}_2 \otimes |n\rangle\langle n| = \frac{\rho^2}{2}\mathbb{I}_{\mathscr{H}_S} + (\widehat{p}\,)^2 \,.$$

and differs from $(\widehat{p}\,)^2$ by a term proportional to the regulator ρ.

CS quantization of the Infinite Well

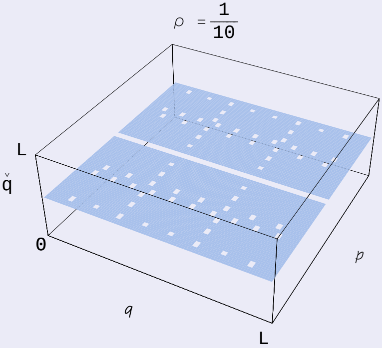
Quantizing the classical observables

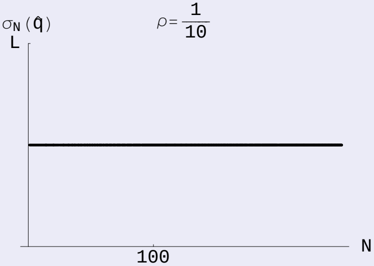
The same procedure might be applied to the position operator:

$$\widehat{q} = \frac{L}{2}\mathbb{I}_{\mathscr{H}_S} - \frac{2L}{\pi^2} \sum_{\substack{n,n' \geq 1, \\ n+n'=2k+1}}^{\infty} e^{-\frac{1}{4\rho^2}(p_n - p_{n'})^2} \left[\frac{1}{(n-n')^2} - \frac{1}{(n+n')^2} \right] \sigma_0 \otimes |n\rangle\langle n'| \, .$$

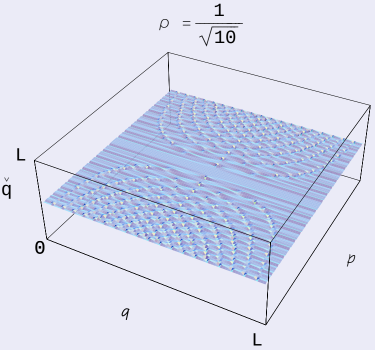
Notice the classical mean value $\frac{L}{2}$ on the diagonal.

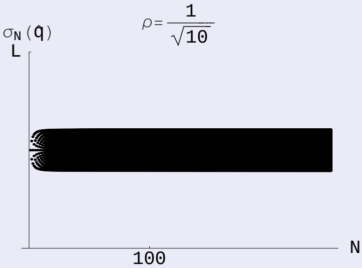
$\langle\hat{q}\rangle(q,p)$, $\sigma(\hat{q})$, $\rho = \frac{1}{10}$



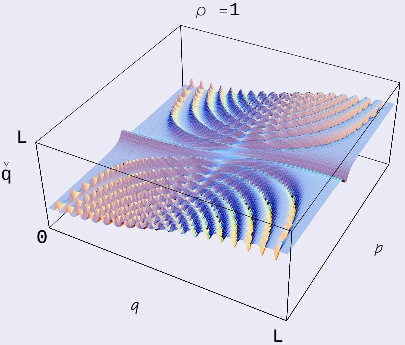


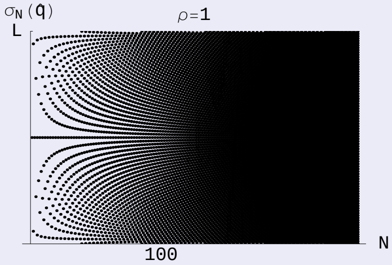
$$\langle\hat{q}\rangle(q,p), \qquad \sigma(\hat{q}), \qquad \rho = \frac{1}{\sqrt{10}}$$





$\langle\hat{q}\rangle(q,p)$, $\sigma(\hat{q})$, $\rho = 1$





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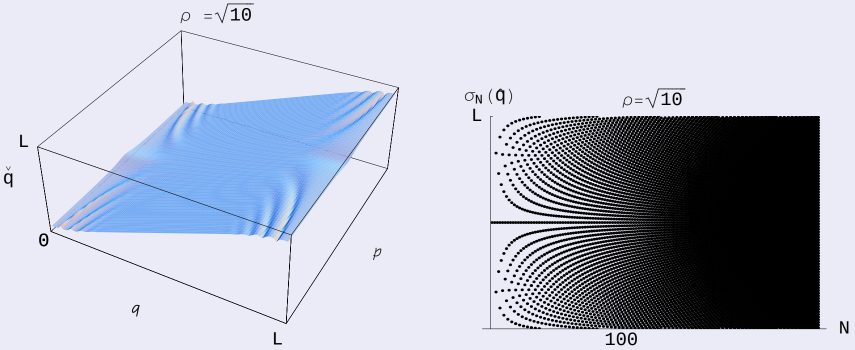
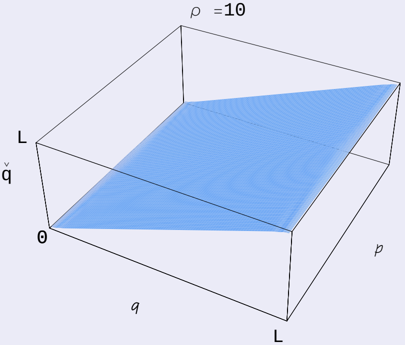
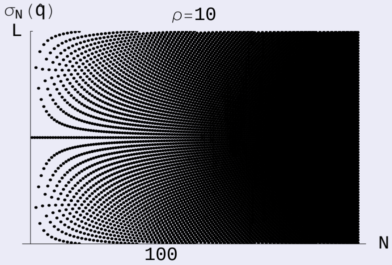


$\langle\hat{q}\rangle(q,p), \quad \sigma(\hat{q}), \quad \rho = 10$





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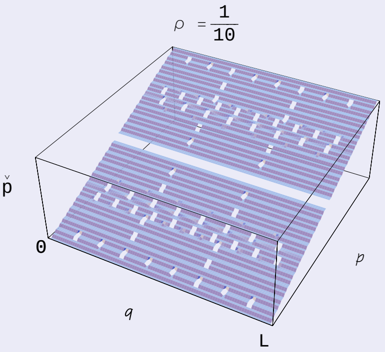

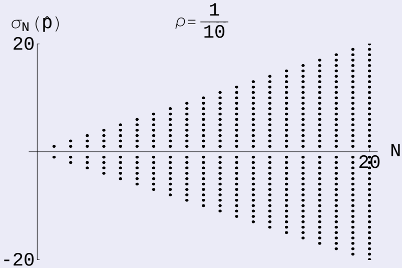

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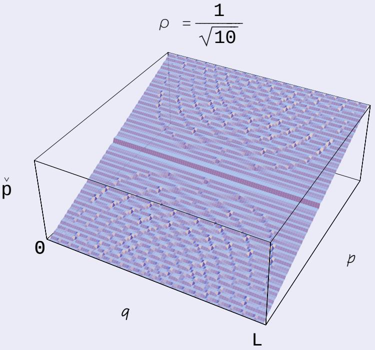

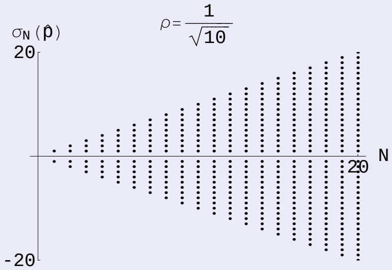

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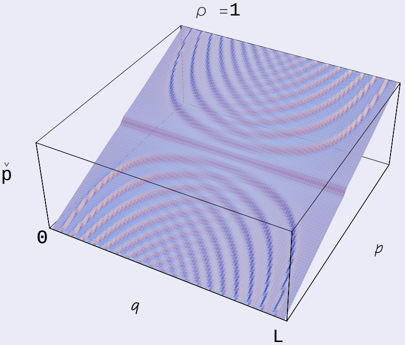

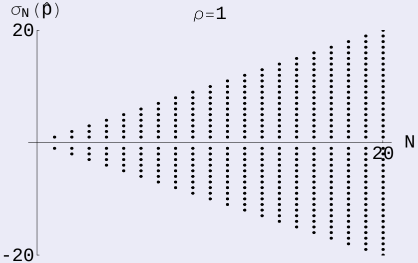

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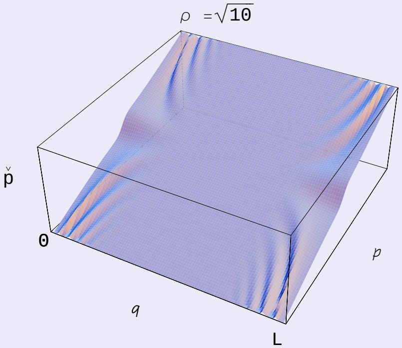

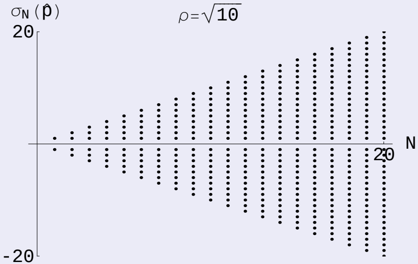

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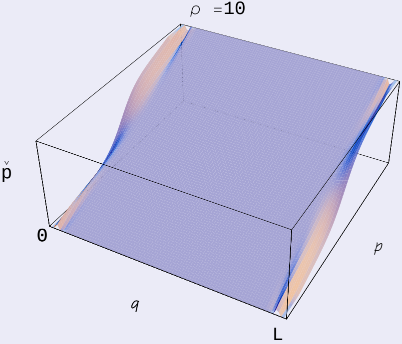
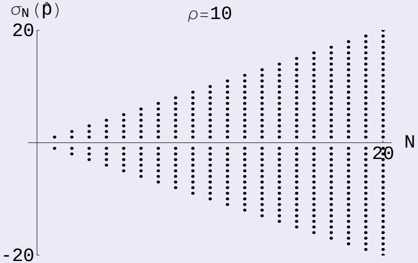
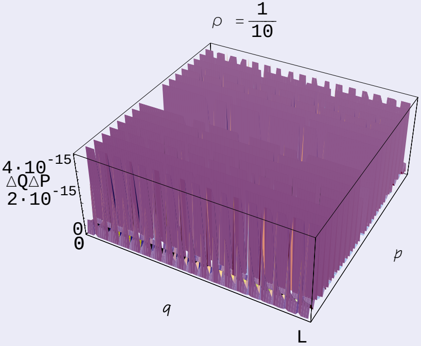
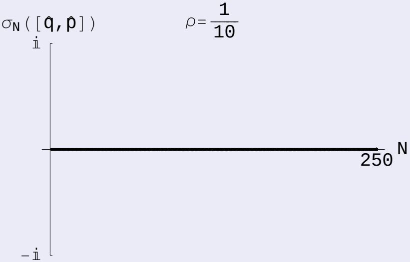




$$\Delta q \Delta p, \quad \sigma([\hat{q},\hat{p}]), \quad \rho = \frac{1}{10}$$





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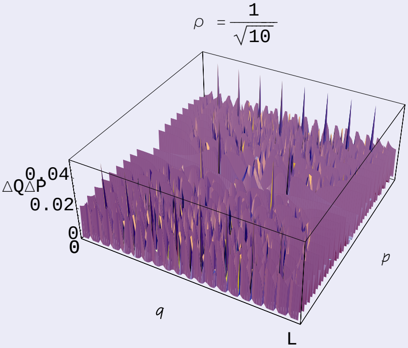

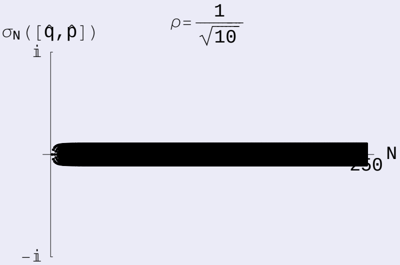

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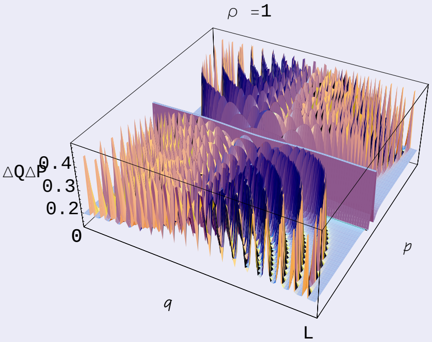

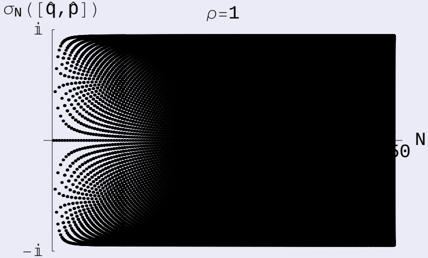

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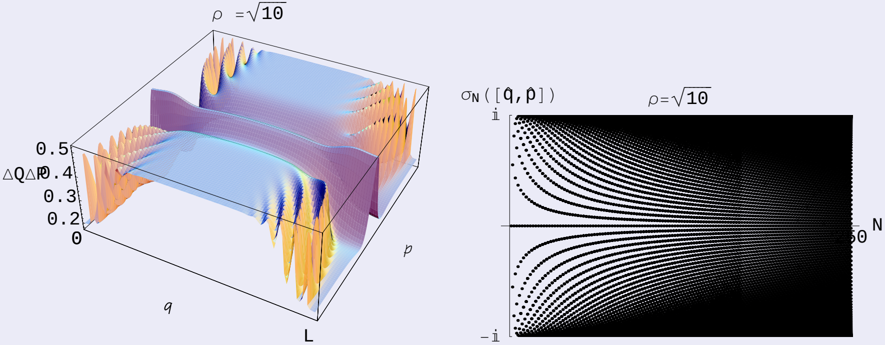
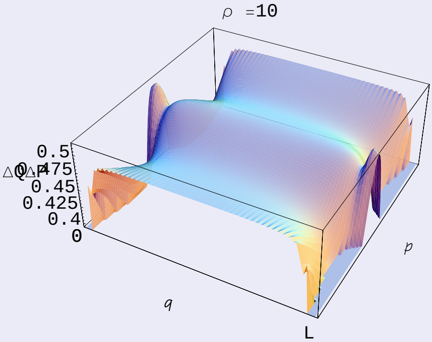
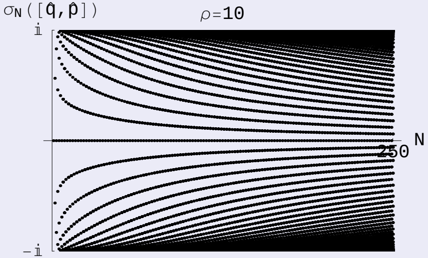


$\Delta q \Delta p$, $\sigma([\hat{q}, \hat{p}])$, $\rho = 10$





CS quantization of the Infinite Well

Final comments

To summarize the results:

- quantizing the free particle trapped in an infinite well is not as simple as it first appears,
- the momentum operator and the Hamiltonian admit an infinite set of self-adjoint extensions (parametrized by $U(1)$ and $U(2)$ respectively),
- the quantization through coherent states construct well behaved operators $\hat{p}$ and $\hat{p^2}$,
- the regulator ρ interpolates between a *classical* behavior and a *quantum* one.

The (fuzzy) sphere (1)

Madore's construction

Consider the decomposition of $f \in C^\infty(S^2)$ in spherical harmonics,

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} \, Y_{\ell m}.$$

Through $S^2 \subset \mathbb{R}^3$, any function in S^2 can be seen as the restriction of a function on $\mathbb{R}^3$. Such functions are generated by the homogeneous polynomials in $\mathbb{R}^3$.

$$f(x) = f_{(0)} + \sum_{(i_1)} f_{(i)} \, x^i + ... + \sum_{(i_1 i_2 \dots i_\ell)} f_{(i_1 i_2 \dots i_\ell)} \, x^{i_1} \, x^{i_2} ... x^{i_\ell} + ...,$$

where each sum subtends a V^ℓ and involves all symmetric combinations of the i_k indices, $i_k = 1, 2, 3$.

This gives, for each fixed value of ℓ, $2\ell + 1$ coefficients $f_{(i_1 i_2 \dots i_\ell)}$ (ℓ fixed), which are those of the symmetric traceless $3 \times 3 \times ... \times 3$ (ℓ times) tensors.

J. Madore: An Introduction to Noncommutative Differential Geometry and its Physical Applications CUP 1995

The (fuzzy) sphere (2)

Madore's construction

The fuzzy sphere with $L+1=2j+1$ cells is usually written $S_{\mathrm{fuzzy},j}$, with j an integer or semi-integer. To obtain it,

I. Consider the three generators J_i of the $2j+1$ dimensional irreducible unitary representations (IUR's) of $SU(2)$. They are $(2j+1)\times(2j+1)$ Hermitian matrices obeying

$$\left[J_i,\, J_j\right] = i\varepsilon_{ijk}J_k.$$

and the Casimir operator is fixed:

$$J_1^2+J_2^2+J_3^2=j(j+1)$$

The (fuzzy) sphere (2)

Madore's construction

II. To define the operator $\widehat{f}$ associated to any function f, replace each coordinate x^i by the $(2j+1)\times(2j+1)$ matrix $\widehat{x^i} \equiv rJ^i$ (r gives a length dimension). Also replace the usual product by the symmetrized matrix product and truncate the sum at index $\ell = 2j$.

III. This associates to the function f the operator $\widehat{f}$, which is an $(2j+1)\times(2j+1)$ matrix.

The (fuzzy) sphere (3)

Madore's construction

The vector space Mat_{2j+1} of $(2j+1)\times(2j+1)$ matrices is linearly generated by a number $(2j+1)^2$ of independent matrices.
These matrices can be taken as the products of the J_i up to power $2j$.
This is necessary and sufficient to close Mat_{2j+1} as an algebra. This provides a finite approximation to the set of functions on the sphere, $^\infty(S^2)$.
This finite *noncommutative* matrix geometry provides a fuzzy approximation to the smooth sphere S^2. The latter appears as the limit $j \to \infty$ of the sequence of these geometries:

$$(\widehat{x^1})^2+(\widehat{x^2})^2+(\widehat{x^3})^2=r^2j(j+1)\xrightarrow[rj=1]{\substack{r\to 0\\ j\to\infty}}(x^1)^2+(x^2)^2+(x^3)^2=1.$$

Application to the $1+1d$ de Sitter spacetime

The general geometry

De Sitter space is seen as a one-sheeted hyperboloid embedded in a three-dimensional Minkowski space:

$$M_H = \{x \in \mathbb{R}^3 \mid x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = (x^0)^2 - (x^1)^2 - (x^2)^2 = -R^2\}$$

$$\alpha, \beta = 0, 1, 2, \quad \eta_{\alpha\beta} = \mathrm{diag}(1, -1, -1)$$

The de Sitter group: $G = SO_0(1,2)$ and its double covering $SU(1,1) \sim SL(2,\mathbb{R})$
Its Lie algebra is spanned by the three Killing vectors:

$$K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha .$$

Application to the $1+1d$ de Sitter spacetime

The group and its IURs

The three Killing vectors are represented as (essentially) self-adjoint operators in an Hilbert space of functions on M_H, square integrable with respect to some invariant inner (Klein-Gordon type) product:

$$K_{\alpha\beta} \longrightarrow L_{\alpha\beta} = M_{\alpha\beta} - i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)$$

K_{12}: compact, for "space translations", K_{02}: non compact, for "time translations", K_{01}: non compact, for Lorentz boosts.
Casimir operator has eigenvalues which determine the UIR's:

$$Q^{(1)} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -j(j+1) = \left(\rho^2 + \frac{1}{4}\right)$$

where $j = -\frac{1}{2} + i\rho$ for the principal serie of $SO_0(1,2)$.

Application to the $1+1d$ de Sitter spacetime

The idea

Comparing both constraints :

- Geometric: $(x^1)^2 + (x^2)^2 - (x^0)^2 = R^2,$
- Group theoretical (in the principal series):

$$Q^{(1)} = M_{01}^2 + M_{02}^2 - M_{12}^2 = \left(\rho^2 + \frac{1}{4}\right)$$

suggests the fuzzy correspondence (introducing r with length dimension):

$$x^\alpha \mapsto \widehat{x^0} = rM_{12},\ \widehat{x^1} = rM_{20},\ \widehat{x^2} = rM_{01}$$

with commutation rules

$$[\widehat{x^1}, \widehat{x^2}] = -ir\widehat{x^0},\ [\widehat{x^0}, \widehat{x^1}] = ir\widehat{x^2},\ [\widehat{x^0}, \widehat{x^2}] = -ir\widehat{x^1},$$

and with

$$\widehat{x^1}^2 + \widehat{x^2}^2 - \widehat{x^0}^2 = r^2\left(\rho^2 + \frac{1}{4}\right)$$

and its “commutative classical limit”: $r \rightarrow 0,\ \rho \rightarrow \infty,\ r\rho = R\,(\text{fixed})$

Application to the $1+1d$ de Sitter spacetime

The idea (continued)

Classical limit is possible with principal series UIR only! They are labelled by parameter $\chi = (j, \epsilon)$.

- $j = -\frac{1}{2} + i\rho$, $\rho \in$, and $\epsilon = 0$ or $\frac{1}{2}$.
- They act in the Hilbert space

$$L^2([0,2\pi)) = \left\{ f(e^{i\theta}) \ : \ \|f\|^2 = \frac{1}{2\pi}\int\limits_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta < \infty \right\}$$

  of the exponential Fourier series.
- The representation operator T^{ps}_χ is given for $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in SU(1,1)$ by:

$$T^{\text{ps}}_\chi(g) f(e^{i\theta}) = \left(\beta e^{i\theta} + \bar{\alpha}\right)^{j+\epsilon} \left(\bar{\beta} e^{-i\theta} + \alpha\right)^{j-\epsilon} f\left(\frac{\alpha e^{i\theta} + \bar{\beta}}{\beta e^{i\theta} + \bar{\alpha}}\right).$$

Application to the $1+1d$ de Sitter spacetime

CS quantization of $1+1d$ dS hyperboloid: first step

- The "observation" set X is the hyperboloid M_H.
- Suitable global coordinates are those of the topologically equivalent cylindrical structure:

$$(\tau,\theta),\ \tau \in \mathbb{R},\ 0 \leq \theta < 2\pi,$$

through the following parametrization,

$$\begin{cases} x^0 = r\tau \\ x^1 = r\tau\cos\theta - r\rho\sin\theta \\ x^2 = r\tau\sin\theta + r\rho\cos\theta. \end{cases}$$

- For the measure on $X = M_H$, we choose the invariant measure (up to a factor) : $\mu(dx) = \frac{1}{2\pi}\, d\tau\, d\theta$.

Application to the $1+1d$ de Sitter spacetime

CS quantization of $1+1d$ dS hyperboloid: second step

- The functions $\phi_m(x)$ forming an ONB needed to construct coherent states are suitably weighted Fourier exponentials:

  $$\phi_m(x) = \left(\frac{\epsilon}{\pi}\right)^{1/4} \, e^{-\frac{\epsilon}{2}(\tau-m)^2} \, e^{im\theta}, \ m \in ,$$

  where $\epsilon > 0$ can be arbitrarily small.
- This parameter ϵ represents a necessary regularization. (Remember the regulator ρ in the infinite well).
- Note that the continuous distribution $x \mapsto |\phi_m(x)|^2$ is the normal law centered at m (for the "time" variable J).

Application to the $1+1d$ de Sitter spacetime

CS quantization of $1+1d$ dS hyperboloid: third step

- Coherent states[1] read

$$|\tau,\theta\rangle = \frac{1}{\sqrt{\mathcal{N}(\tau)}} \left(\frac{\epsilon}{\pi}\right)^{1/4} \sum_{m\in\mathbb{Z}} e^{-\frac{\epsilon}{2}(\tau-m)^2}\, e^{-im\theta}|m\rangle,$$

- The normalization factor

$$\mathcal{N}(x) \equiv \mathcal{N}(\tau) = \sqrt{\frac{\epsilon}{\pi}} \sum_{m\in\mathbb{Z}} e^{-\epsilon(\tau-m)^2} < \infty$$

  is a periodic train of normalized Gaussians and is proportional to an elliptic Theta function.

- Applying the Poisson summation yields the alternative form :

$$\mathcal{N}(\tau) = \sum_{m\in\mathbb{Z}} e^{2\pi i m\tau}\, e^{-\frac{\pi^2}{\epsilon}m^2}.$$

  From this formula we easily prove that $\lim_{\epsilon\to 0}\mathcal{N}(\tau) = 1$.

---

[1] These coherent states have been proposed by De Bièvre-González (1992-93), González-Del Olmo (1998), Kowalski-Rembieliński-Papaloucas (1996)

Application to the $1+1d$ de Sitter spacetime

CS quantization of $1+1d$ dS hyperboloid: fourth step

- With the CS quantization scheme, the quantum operator (acting on $\mathscr{H}_S$) associated to the classical observable $f(x)$ is obtained by

$$\hat{f} = A_f := \int_X f(x)|x\rangle\langle x|\mathcal{N}(x)\mu(dx).$$

- For the most basic one, associated to the coordinate τ

$$\hat{\tau} = A_\tau = \sum_{m\in\mathbb{Z}} m|m\rangle\langle m|,$$

- This operator reads in angular position representation (Fourier series):

$$A_\tau = -i\frac{\partial}{\partial\theta},$$

  and easily identifies to the compact representative M_{12} of the Killing vector X_{12} in the principal series UIR.
- Thus, "time" dS component x^0 is naturally quantized, with spectrum $r\mathbb{Z}$ through $x^0 \mapsto \widehat{x^0} = rM_{12}$

Application to the $1+1d$ de Sitter spacetime

CS quantization of $1+1d$ dS hyperboloid: fifth step

- For the ambient coordinates

$$\widehat{x^0} = r \sum_{m\in\mathbb{Z}} m|m\rangle\langle m|$$

$$\widehat{x^1} = r\frac{e^{-\frac{\epsilon}{4}}}{2} \sum_{m\in\mathbb{Z}} \left\{\left(m+\frac{1}{2}+i\rho\right)|m+1\rangle\langle m| + h.c\right\}$$

$$\widehat{x^2} = r\frac{e^{-\frac{\epsilon}{4}}}{2i} \sum_{m\in\mathbb{Z}} \left\{\left(m+\frac{1}{2}+i\rho\right)|m+1\rangle\langle m| - h.c\right\}$$

- Commutation rules are those of $so(1,2)$ up to a global (r) factor and a local ($e^{\epsilon/2}$) factor:

$$[\widehat{x}_0, \widehat{x}_1] = ir\widehat{x}_2, \qquad [\widehat{x}_0, \widehat{x}_2] = -ir\widehat{x}_1, \qquad [\widehat{x}_1, \widehat{x}_2] = -ire^{-\epsilon/2}\widehat{x}_0.$$

- Commutative limit at $r \to 0$ (and $\rho \to \infty$) is apparent.

Application to the $1+1d$ de Sitter spacetime

quadratic expressions

$$\widehat{x^0x^0} = \widehat{x^0}\,\widehat{x^0} + \frac{r^2}{2\epsilon}\mathbb{I}$$

$$\widehat{x^0x^1} = \frac{1}{2}\left\{\widehat{x^0},\, \widehat{x^1}\right\} + \frac{r^2}{2\epsilon}\widehat{\cos\theta}$$

$$\widehat{x^0x^2} = \frac{1}{2}\left\{\widehat{x^0},\, \widehat{x^2}\right\} + \frac{r^2}{2\epsilon}\widehat{\sin\theta}$$

$$\widehat{x^1x^1} = \cosh\left(\frac{\epsilon}{2}\right)\widehat{x^1}\,\widehat{x^1} + \sinh\left(\frac{\epsilon}{2}\right)\widehat{x^2}\,\widehat{x^2} + \frac{r^2}{2\epsilon}\widehat{\cos^2\theta} - \frac{r^2}{4}\widehat{\sin^2\theta}$$

$$\widehat{x^1x^2} = \frac{e^{-\frac{\epsilon}{2}}}{2}\left\{\widehat{x^1},\, \widehat{x^2}\right\} + \frac{r^2}{2}\left(\frac{1}{2\epsilon} + \frac{1}{4}\right)\widehat{\sin(2\theta)}$$

$$\widehat{x^2x^2} = \sinh\left(\frac{\epsilon}{2}\right)\widehat{x^1}\,\widehat{x^1} + \cosh\left(\frac{\epsilon}{2}\right)\widehat{x^2}\,\widehat{x^2} + \frac{r^2}{2\epsilon}\widehat{\sin^2\theta} - \frac{r^2}{4}\widehat{\cos^2\theta}$$

Quantum corrections vanish at the limit $r \rightarrow 0$ ($\rho \rightarrow \infty$), "continuum limit".

Application to the $1+1d$ de Sitter spacetime

more general expressions

Pick

$$n = (n^0, n^1, n^2), n' = (n'^0, n'^1, n'^2) \in^3 .$$

Then

$$\widehat{e^{(n+n')\cdot x}} = \widehat{e^{n\cdot x}}\,\widehat{e^{n'\cdot x}} + o(r)$$

Same conclusion for the commutative limit at $r \rightarrow 0$.
This defines pertubatively a Moyal product.

From $1+1d$ to $1+3d$

The 1+1 construction is extended in the following way

$\boxed{1+1\ d}$

the group :

$\widetilde{SO_0(1,2)} = SU(1,1)$,
$\dim(\mathfrak{o}(1,2) = \mathrm{span} M_{\alpha\beta}) = 3$,

whose Casimir's are

$$Q^{(1)} = M_{01}^2 + M_{02}^2 - M_{12}^2 = (\rho^2 + \tfrac{1}{4})\mathbb{I}$$

in the principal serie, labeled by
$(j = -\frac{1}{2} + i\rho,\ \epsilon = 0 \text{ or } \frac{1}{2}),\ \ \rho \in \mathbb{R}^+$
suggest the following correspondences
$x^0 \mapsto M_{12}, \ldots$

$\boxed{1+3\ d}$

$\widetilde{SO_0(1,4)} = Sp(2,2)$,
$\dim(\mathfrak{o}(1,4) = \mathrm{span} L_{\alpha\beta}) = 10$,

$$Q^{(1)} = -\tfrac{1}{2} L_{\alpha\beta} L^{\alpha\beta} = \left(\tfrac{9}{4} + \nu^2 - s(s+1)\right) \mathbb{I}$$

$$Q^{(2)} = -W_\alpha W^\alpha = \left(\tfrac{1}{4} + \nu^2\right) s(s+1)\mathbb{I}$$
$$W_\alpha := -\tfrac{1}{8} \varepsilon_{\alpha\beta\gamma\delta\epsilon} L^{\beta\gamma} L^{\delta\epsilon}$$

$(q = \frac{1}{2} + i\nu,\ s)$ with $\nu \in \mathbb{R}^+$, $s \in \mathbb{N}/2 \approx$ sp

$x^\alpha \mapsto W^\alpha$

From $1+1d$ to $1+3d$

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$(q = \frac{1}{2} + i\nu,\ s)$ with $\nu \in \mathbb{R}^+$, $s \in \mathbb{N}/2 \approx$ sp

$x^\alpha \mapsto W^\alpha$

From 1+1 d to 1+3 d

The ideas behind the $1+1d$

To quantize τ we are looking for eigenfunctions (with a discrete spectrum) of its corresponding operator.
M_{12} Casimir of $SO(2)$ (compact) leads us to solve the eigenvalue problem :

$$M_{12} f_m(\theta) = m f_m(\theta),\ \theta \in [0,\, 2\pi).$$

Here $f_m(\theta) = e^{im\theta} \in L^2_{\mathbb{C}}(S^1)$ (as expected). Then, our choice of coherent states is

$$|\tau,\, \theta\rangle = \frac{1}{\sqrt{\mathcal{N}(\tau)}} \left(\frac{\epsilon}{\pi}\right)^{\frac{1}{4}} \sum_m e^{-\frac{\epsilon}{2}(\tau-m)^2} \overline{f_m}(\theta) |m\rangle.$$

From $1+1d$ to $1+3d$

Applied to the $1+3d$ case

To quantize τ we are looking for eigenfunctions (with a discrete spectrum) of its corresponding operator.
W^0 a Casimir of $SO(4) \sim SU(2)_L \times SU(2)_R$ (compact)

$$W^0 \mathcal{Z}_{\tau_i \mathcal{I}}(\xi) = \tau_i \mathcal{Z}_{\tau_i \mathcal{I}}(\xi),\ \xi \in SU(2) \sim S^3$$

where

$$\mathcal{Z}_{\tau_i\, \mathcal{I}}(\xi) = \mathcal{Z}^{s j_L\ j_R}_{\quad m_L\, m_R}(\xi) = \sum_{\sigma,\mu} \sqrt{2j_R+1} \begin{pmatrix} s & j_L & j_R \\ \sigma & \mu & -m_R \end{pmatrix} e^s{}_\sigma \otimes \sqrt{\frac{2j_L+1}{2\pi^2}} D^{j_L}_{m_L \mu}(\xi),$$

$$\mathcal{Z}_{\tau_i\, \mathcal{I}}(\xi) \in L^2_{2s+1}(S^3).$$

One has

$$W^0 \mathcal{Z}^{s j_L\ j_R}_{\quad m_L\, m_R} = (j_L(j_L+1) - j_R(j_R+1)) \mathcal{Z}^{s j_L\ j_R}_{\quad m_L\, m_R}, \quad |j_L - s| \leq j_R \leq j_L + s$$

It is precisely the orthonormal set which is to used in the construction of coherent states in view of the construction of the $1+3d$ fuzzy de Sitter hyperboloid.

Application to the $1+3d$ Spacetime

The general geometry

- dS: one-sheeted hyperboloid embedded in a five-dimensional Minkowski space:

$$M_H \equiv \{x \in \mathbb{R}^5 \,|\, x^2 = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\}$$

$$\alpha, \beta = 0, 1, 2, 3, 4, \;\; \eta_{\alpha\beta} = diag(1, -1, -1, -1, -1)$$

- Lie algebra: ten Killing vectors

$$K_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha .$$

- Universal covering of $SO_0(1,4)$ is the symplectic $Sp(2,2)$ group (needed for half-integer spins) : subgroup of the group of 2×2 matrices with quaternionic coefficients:

$$Sp(2,2) = \left\{ g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) a, b, c, d \in \mathbb{H} \simeq \mathbb{R}^+ \times SU(2), \; \det_{4\times 4} g = 1, \;\; g^\dagger \gamma^0 g = \gamma^0 \right\}$$

$g^\dagger = \bar{g}^t$: quaternionic conjugate and transpose of g, $\gamma^0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$

Application to the $1+3d$ Spacetime

dS Unitary Irreducible Representations (UIR): Lie algebra level

- Quantization (geometrical or Berezin coherent state or something else) of de Sitter classical phase spaces leads to their quantum counterparts, namely the quantum elementary systems associated in a biunivocal way to the UIR's of the de Sitter group $Sp(2,2)$
- The ten Killing vectors are represented as (essentially) self-adjoint operators in Hilbert space of (spinor-)tensor valued functions on M_H, square integrable with respect to some invariant inner (Klein-Gordon type) product :

$$K_{\alpha\beta} \rightarrow L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta},$$

  $M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha)$ (orbital part) $S_{\alpha\beta}$ (spinorial part) acts on indices of functions in a certain permutational way
- Two Casimir operators whose eigenvalues determine the UIR's :

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta} \qquad Q^{(2)} = -W_\alpha W^\alpha, \;\; W_\alpha = -\frac{1}{8} \epsilon_{\alpha\beta\gamma\delta\eta} L^{\beta\gamma} L^{\delta\eta}.$$