A (conformally covariant) quantization of the Maxwell field on de Sitter spacetime

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Outline of the talk

- Introduction
- A brief reminder on conformal invariance
- Geometry in $\mathbb{R}^6$
- Quantizing the scalar field
- Quantizing the Maxwell field
- Conclusion & Outlooks

Bibliography:

- S. Faci, E. Huguet, JQ, J. Renaud, PRD 80, 124005(2009).
- E. Huguet, JQ, J. Renaud, PRD 77, 044025(2008).
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- JQ, Chap.3 PhD Thesis (mostly).

Introduction

de Sitter spacetime?

- Observations on far away supernovæ: $\Lambda > 0$,
- Inflation(s),

- Maximally symmetric (10 param. group of isometries),
- Globally hyperbolic,
- The *simplest* curved spacetime generalization of Minkowski.

But ... *no* Killing vector which is timelike everywhere.

[(time?, asymptotic states?, ...)]

Introduction

Quantum fields? 1/3

On Minkowski spacetime a QFT fulfills Gårding-Wightman axioms:

1. Relativity,
2. Spectral condition,
3. Vacuum state,
4. Vacuum compatibility and invariant domains,
5. Field regularity,
6. Covariance,
7. Microcausality,
8. Completeness.

Introduction

Quantum fields? 2/3

Let us recall the Wightman function:

$$\mathscr{W}^{(n)}(x_1,\ x_2,\ \ldots,\ x_n) = \langle \Omega, \hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_n)\Omega \rangle.$$

For *free* fields $\mathscr{W}^{(n\geq 2)}$ is entirely determined by $\mathscr{W}^{(2)}$ (Wick formula).

[more the def. of quasi-free states]

Then all of the 8 axioms might be recast on the two point function.

Introduction

Quantum fields? 3/3

(a) *Covariance*: $\forall g \in \mathscr{P}_+^\uparrow$:

$$\mathscr{W}^{(2)}(x_1, x_2) = \mathscr{W}^{(2)}(g.x_1, g.x_2).$$

(b) *Spectral condition*: $\widetilde{W}(q_1) = 0$ if $q_1 \notin \overline{V}^+$:

$$\widetilde{\mathscr{W}}^{(2)}(p_1, p_2) = (2\pi)^4 \delta(p_1 + p_2) \widetilde{W}(p_1).$$

(c) *Hermiticity* :

$$\mathscr{W}^{(2)}(x_1, x_2) = \overline{\mathscr{W}^{(2)}(x_2, x_1)}.$$

(d) *Local commutation*: for $(x_1 - x_2)^2 < 0$:

$$\mathscr{W}^{(2)}(x_1, x_2) = \mathscr{W}^{(2)}(x_2, x_1).$$

(e) *Positivity*: $\forall f_i \in \mathscr{S}(\mathbb{R}^4)$:

$$\iint \overline{f(x_1)} \mathscr{W}^{(2)}(x_1, x_2) f(x_2) \, \mathrm{d}x_1 \mathrm{d}x_2 \geq 0.$$

Introduction

Comments

On dS this axiomatic can be used "point by point" [Bros, Moschella]

However, due to the lack of a globally timelike Killing vector there is another criterion to use in order to select the Hilbert space carrying the CCR rep.

Here this will be the *conformal energy* positivity condition $\sigma(\hat{X}_{05}) \geq 0$.

Otherwise, on its Minkowskian realization this would lead to negative energies (in direct conflict with Gårding-Wightman axioms).

J. Bros and U. Moschella, Rev. Math. Phys, **8***:327-392, (1996). arXiv:gr-qc/9511019*

Introduction

What we would like to achieve

A QED on de Sitter spacetime.

- We already have the scalar field, spinorial field, (massive) vector and rank 2 tensor, ... [Bros, Gazeau, Moschella, PhD minions]

Mainly two difficulties to overcome:

1. Make sense of interaction(s) [Bros, Epstein, Moschella]
   Not the subject of this talk.
2. The Maxwell field was, of course, quantized previously [Dimock, Allen & Jacobson, Higuchi, Woodard, ...] however it didn't quite fit in our formalism. Moreover we wanted to:
   - keep track of the action of $SO_0(1,4)$ and $SO_0(2,4)$,
   - follow the null curvature limit in the simplest fashion possible.

J. Bros, U. Moschella, J-P. Gazeau, Phys. Rev. Lett., **73***:1746-1749,*
P. Bartesaghi, J-P. Gazeau, U. Moschella, M.V. Takook, Class. Quant. Grav., **18***:4373-4394, (2001). etc*

J. Bros, H. Epstein and U. Moschella, JCAP **0802** *(2008) 003 arXiv:hep-th/0612184,*
J. Bros, H. Epstein and U. Moschella, Annales Henri Poincare **11** *(2010) 611, arXiv:0812.3513,*
S. Hollands, arXiv:1010.5367

J. Dimock, Rev. Math. Phys. **4***:223-233, (1992),*
B. Allen and T. Jacobson, Comm. Math. Phys. **103***:669, (1986), etc.*

Introduction

Subject of the talk

Quantize the conformal scalar field (mccf) and Maxwell's field.

The scalar field:

- no gauge invariance,
- enables us to set our framework in $\mathbb{R}^6$.

The maxwell field:

- coping with the gauge invariance → a Gupta-Bleuler scheme,
- re-use the whole technology from the scalar field.

Rmk. Througout the whole presentation $d = 4$.
Rmk. By Maxwell field we mean the 4-potential A^μ, as it is the one with which the interaction computations are made.

A brief reminder on conformal invariance

Weyl transformations (GR)

On the one hand consider the Weyl transformations (rescalings):

$$(M, g) \mapsto (M, \overline{g}) \text{ where } \overline{g}_{\mu\nu}(x) = \omega^2(x) g_{\mu\nu}(x).$$

Def. $E_g(F) = 0$ is *Weyl invariant*, if $\exists s \in \mathbb{R}$ st:

$$E_{\overline{g}}(\overline{F}) = \omega^t E_g(F) = 0,$$

where $\overline{F} = \omega^s F$ and $t \in \mathbb{R}$.

(s is the conformal weight of the field)

A brief reminder on conformal invariance

Weyl transformations (GR)

The equation of the conformal scalar (Yamabe's equation):

$$\Big(\Box - \frac{R}{6}\Big)\phi = 0, \qquad \overline{\phi} = \omega^{-1}\phi,$$

and Maxwell's equations (in vacuum):

$$\begin{aligned} \nabla_\mu F^{\mu\nu} &= 0, & F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu, \\ \overline{A}_\mu &= A_\mu, & \overline{A}^\mu &= \overline{g}^{\mu\nu}\overline{A}_\nu = \omega^{-2}A^\mu, \end{aligned}$$

are Weyl invariant.

This is exactly what makes the null curvature limit of our results trivial and enables us to relate the space of solutions (*cf.* later).

A brief reminder on conformal invariance

Conformal group (PΦ)

On the other hand, for (M, g) lorentzian the group leaving the causality invariant (light cone) will be called the *conformal group.*

Rmk. The isometries of (M, g) obviously belong to the conformal group.

In the case of the Minkowski and of the de Sitter spacetime the conformal group is locally isomorphic to $SO_0(2, 4)$.

Prop. (Characterization). The generators X of the conformal group fulfill the conformal Killing equation:

$$\mathcal{L}_X g = 2 f_X(x) g,$$

$\mathcal{L}_X$: Lie derivative along X, $f_X(x)$: function on the spacetime.

A brief reminder on conformal invariance

Conformal group (PΦ)

Def. An equation $E_g(F) = 0$ will be *conformally invariant* if one can realize the Lie algebra $\mathfrak{g}$ of the conformal group $SO_0(2,4)$ such that

$$[E_g, \mathfrak{g}](F) = \zeta E_g(F),$$

where ζ is a function.

Then the space of solutions of $E_g(F) = 0$ is closed under $SO_0(2,4)$.

A brief reminder on conformal invariance

Conformal group (PΦ)

Fact. the space of solutions of the conformal scalar and $\mathscr{V}_{\mathscr{M}}$ the space of solutions of Maxwell's equations, on dS/Mink., is closed under the action of $SO_0(2,4)$ once one has taken into account the presence of a multiplier:

$$[T_g\phi](x) = (\alpha(g, g^{-1}.x))^{-1}\phi(g^{-1}.x), \qquad \forall g \in SO_0(2,4),$$

where the multiplier α is defined by:

$$\mathrm{d}g.s^2 = (\alpha(g,x))^2\mathrm{d}s^2$$

and fulfills, by construction, the equation:

$$\alpha(gg', x) = \alpha(g, g'.x)\alpha(g', x).$$

Geometry in $\mathbb{R}^6$

The null cone $\mathcal{C}$ in $\mathbb{R}^6$ and the submanifold X_ξ

$\mathbb{R}^6$ is the natural playground for $SO_0(2,4)$. [Dirac, ...]

Consider the null cone $\mathcal{C}$ left invariant by $SO_0(2,4)$:

$$\mathcal{C} = \left\{ z \in \mathbb{R}^6 : \eta_{\alpha\beta} z^\alpha z^\beta = (z^5)^2 + (z^0)^2 - \|\mathbf{z}\|^2 - (z^4)^2 = 0 \right\},$$

the hyperplane:

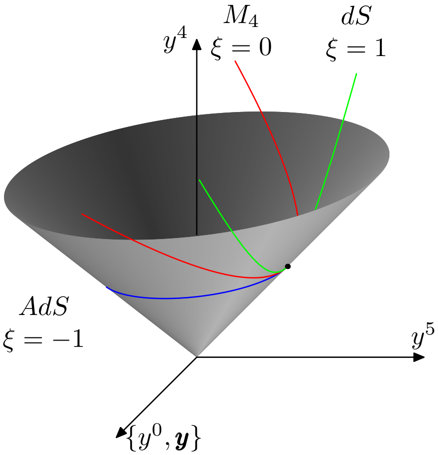
$$P_\xi = \left\{ z \in \mathbb{R}^6 : (1+\xi) z^5 + (1-\xi) z^4 = 2 \right\}, \quad \xi \in \mathbb{R},$$

and finally the 4-dimensional submanifold:

$$X_\xi = \mathcal{C} \cap P_\xi .$$

Geometry in $\mathbb{R}^6$

The null cone $\mathcal{C}$ in $\mathbb{R}^6$ and the submanifold X_ξ



Geometry in $\mathbb{R}^6$

The null cone $\mathcal{C}$ in $\mathbb{R}^6$ and the submanifold X_ξ

In the ambiant $\mathbb{R}^6$ space the $\mathfrak{o}(2,4)$ algebra is spanned by the 15 vectors:

$$X_{\alpha\beta} = z_\alpha \partial_\beta - z_\beta \partial_\alpha, \qquad \alpha, \beta = 5, 0, 1, 2, 3, 4.$$

The submanifold X_ξ is left invariant by the 10 vectors $\{X_{\mu\nu}, Y^\xi_\mu\}$ where:

$$2Y^\xi_\mu = [(1+\xi)X_{\mu 4} - (1-\xi)X_{\mu 5}], \qquad \mu, \nu = 0, 1, 2, 3.$$

A straightforward calculation leads to:

$$\begin{aligned} [X_{\mu\nu}, X_{\rho\sigma}] &= \ -(\eta_{\mu\rho}X_{\nu\sigma} + \eta_{\nu\sigma}X_{\mu\rho} - \eta_{\mu\sigma}X_{\nu\rho} - \eta_{\nu\rho}X_{\mu\sigma}), \\ [Y^\xi_\mu, Y^\xi_\nu] = \xi X_{\mu\nu}, & \qquad\quad [Y^\xi_\rho, X_{\mu\nu}] = \eta_{\mu\rho} Y^\xi_\nu - \eta_{\nu\rho} Y^\xi_\mu, \end{aligned}$$

which are the familiar algebra of dS/Poincaré/AdS group.

Varying continuously ξ deforms the above algebra in $\mathfrak{o}(2,4)$.

Geometry in $\mathbb{R}^6$

The null cone $\mathcal{C}$ in $\mathbb{R}^6$ and the submanifold X_ξ

In addition, the induced (lorentzian) metric $g_\xi = \eta|_{X_\xi}$ on X_ξ has a constant curvature scalar $R_\xi = -12\xi$.

With the above mentioned maximal symmetry of X_ξ one has:

$$(X_\xi, g_\xi = \eta|_{X_\xi}) = \begin{cases} \xi = H^2 & : \text{dS}, \\ \xi = 0 & : \text{Mink}, \\ \xi = -H^2 & : \text{AdS}. \end{cases}$$

Rmk. This encompasses the *whole* dS/Mink/AdS spacetime.
Rmk. H will then be the Hubble function of dS/AdS which is a constant.

Geometry in $\mathbb{R}^6$

The cone up to dilation $\mathcal{C}'$

Notice that $\mathcal{C}$ has a natural structure of bundle:

$$\mathcal{C} = (S^1 \times S^3) \times \mathbb{R}^+ = \mathcal{C}' \times \mathbb{R}^+.$$

The submanifold $\mathcal{C}'$ has an induced (lorentzian) metric $g_{\mathcal{C}'} = \eta|_{\mathcal{C}'}$.

The points of X_ξ might be projected along the fiber on $\mathcal{C}'$. The metric elements of g_ξ are related to those of $g_{\mathcal{C}'}$ through a Weyl rescaling:

$$g_\xi = (\omega_\xi)^2 g_{\mathcal{C}'},$$

where the Weyl factor is nothing but the ratio of the radius/height between $\mathcal{C}'$ and X_ξ.

Geometry in $\mathbb{R}^6$

The cone up to dilation $\mathcal{C}'$

On $\mathcal{C}'$, seen as the submanifold $S^1 \times S^3$, a global system of coordinates is:

$$\begin{cases} z^5 = \cos\beta, \\ z^0 = \sin\beta, \\ z^i = \sin\alpha\, n^i, \\ z^4 = \cos\alpha, \end{cases}$$

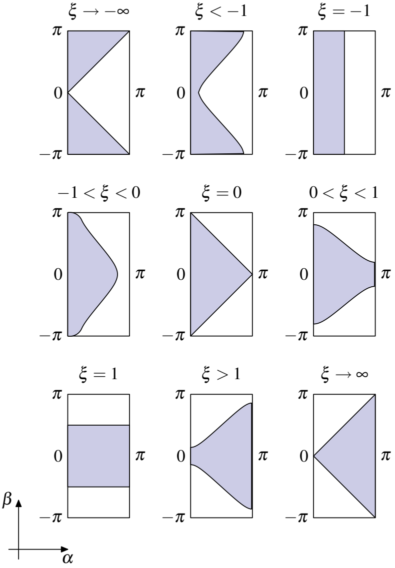
where $\beta \in]-\pi, \pi[$, $\alpha \in [0, \pi]$ and $n \in S^2 \subset \mathbb{R}^3$.

In those the conformal factor ω_ξ reads:

$$\omega_\xi(\alpha, \beta) = \frac{2}{(1-\xi)\cos\alpha + (1+\xi)\cos\beta}.$$

Geometry in $\mathbb{R}^6$

The manifolds X_ξ seen as subsets of $\mathcal{C}'$



The set X_ξ, shaded region, as a subset of the cone up to dilations $\mathcal{C}'$. The angular coordinates on S^2 are suppressed.
1-4: AdS, 5: Mink., 6-9: dS.

Geometry in $\mathbb{R}^6$

Homogeneous fields

- Consider a scalar field ϕ defined in $\mathbb{R}^6$ homogeneous of degree r.
- On the null cone $\mathcal{C}$ the value of the restriction $\phi_{\mathcal{C}'}$ of ϕ can be recovered anytime thanks to the homogeneity:

  $$\phi_{\mathcal{C}'}(x \in \mathcal{C}') = \pi_{\mathcal{C}'}(\phi)(z \in \mathcal{C} \subset \mathbb{R}^6) = \left(\frac{c}{f_{\mathcal{C}'}(z)}\right)^r \phi(z),$$

  with $f_{\mathcal{C}'} = c$ the equation, homogeneous degree 1, defining $\mathcal{C}'$.
- *Lifting* the field ϕ from, say, $\mathcal{C}'$ to X_ξ is *exactly* performing the Weyl transform on the field with conformal weight r.
- For T_g the natural action on $\mathcal{C}$ of $SO_0(2,4)$ one recovers the appropriate action $T_g^{\mathcal{C}'}$ on $\mathcal{C}'$ *including $\alpha(g,x)$* through:

  $$T_g^{\mathcal{C}'} = i^*_{\mathcal{C}'} \circ \pi_{\mathcal{C}'} \circ T_g \circ i_{\mathcal{C}'},$$

  with $i_{\mathcal{C}'}$ the injection of $\mathcal{C}'$ in $\mathcal{C}$.

Geometry in $\mathbb{R}^6$

Homogeneous fields (comments)

Important. We defined the projection on $\mathcal{C}'$ due to the fact that any point in $\mathcal{C}$ can be projected on $\mathcal{C}'$.

The same can be done for X_ξ however global properties have to be taken into account → going to the compactification.

In our case we do not have to go through such subtleties (see hereafter).

Quantizing the scalar field

The equation of motion

Now in $\mathbb{R}^6$ one can consider the field equation $\Box_6 \phi$ and ask that

$$(\Box_6 \phi)|_{\mathcal{C}} = 0, \qquad\qquad z \cdot \frac{\partial}{\partial z}\phi = -\phi.$$

Rmk. The restriction of $\Box_6$ to $\mathcal{C}$ is meaningless ($\det(\eta|_{\mathcal{C}}) = 0$). However restricting its result when applied to field of degree -1 is meaningful [Dirac, ...]

Taking into account the above leads the restriction/projection of ϕ on X_ξ or $\mathcal{C}'$ to fulfill the massless conformally coupled equation which reads:

$$\begin{aligned}(\Box_{\mathcal{C}'} + 1)\phi_{\mathcal{C}'} &= 0 \quad \text{on } (\mathcal{C}', g_{\mathcal{C}'}),\\ (\Box_\xi + 2\xi)\phi_\xi &= 0 \quad \text{on } (X_\xi, g_\xi) \simeq (\mathcal{C}'|_{\omega_\xi > 0}, (\omega_\xi)^2 g_{\mathcal{C}'}).\end{aligned}$$

P.A.M. Dirac, Ann. of Maths., **36***:429-442, (1935).*

G. Mack and A. Salam, Ann. of Phys., **53***:174-202, (1969). etc. (especially Mayer's work in the mid-70)*

Quantizing the scalar field

Solving the equation on $(\mathcal{C}', g_{\mathcal{C}'})$

In the previously used coordinates on $\mathcal{C}'$ one has:

$$\Box_{\mathcal{C}'} = \frac{\partial^2}{\partial\beta^2} - \Delta_{S^3}.$$

Then, a set of modes of positive conformal energy is found:

$$\phi^{\mathcal{C}'}_{Llm} = \frac{1}{\sqrt{2(L+1)}} e^{-i(L+1)\beta} \mathcal{Y}_{Llm}(\alpha, \theta, \varphi),$$

with $\mathcal{Y}$ the hyperspherical harmonics on S^3.
They fulfill the positive conformal energy criterion with:

$$X_{05}\phi^{\mathcal{C}'}_{Llm} = (L+1)\phi^{\mathcal{C}'}_{Llm}, \qquad \sigma(X_{05}) \geq 1.$$

The field is identified to belong to the uir $\mathscr{C}^{>}(1,0,0)$ of $SO_0(2,4)$.
[Bayen *et al.*, Angelopoulos *et al.*, ...]

Quantizing the scalar field

The scalar product and Hilbert space

Then, one can pick a Cauchy surface at $\beta = 0$ and define:

$$\langle \phi_1^{\mathcal{C}'}, \phi_2^{\mathcal{C}'} \rangle_{\mathcal{C}'} = i \int_{\beta=0} \phi_1^{*\mathcal{C}'} \overleftrightarrow{\partial_\beta} \phi_2^{\mathcal{C}'} \, \mathrm{dVol}(S^3).$$

- The modes $\phi_{Llm}^{\mathcal{C}'}$ are orthonormalized wrt the sp.
- The representation (including α) of $SO_0(2,4)$ is unitary wrt the sp.

Then, one can consider the Hilbert space (one particle sector):

$$\mathscr{H}_{\mathcal{C}'} = \left\{ \phi^{\mathcal{C}'} = \sum_{Llm} c_{Llm} \phi_{Llm}^{\mathcal{C}'} : c_{Llm} \in \ell^2(\mathbb{C}) \right\}.$$

Quantizing the scalar field

Going to dS/Mink

Notice the important fact that the hypersurface at $\beta = 0$ is a common Cauchy surface for $\mathcal{C}'$ and X_ξ with $\xi = H^2 \geq 0 \rightarrow$ rules out AdS.

In addition, lifting/rescaling the field to dS is found to be unitary wrt the scalar product:

$$\langle \phi_1^{\mathcal{C}'}, \phi_2^{\mathcal{C}'} \rangle_{\mathcal{C}'} = \langle \phi_1^H, \phi_2^H \rangle_H,$$

with $\phi^H = (\omega_H)^{-1} \phi^{\mathcal{C}'}$ and $\langle \cdot, \cdot \rangle_H$ is the (usual) Klein-Gordon scalar product on dS/Mink.

Then, out of the modes ϕ^H_{Llm} one constructs:

$$\mathscr{H}_H = \left\{ \phi^H = \sum_{Llm} c_{Llm} \phi^H_{Llm} : c_{Llm} \in \ell^2(\mathbb{C}) \right\}.$$

Quantizing the scalar field

The rescaling as an unitary map (1/2)

The lifting/rescaling of the field might then be reinterpreted as a bijective, unitary map between Hilbert spaces:

$$\begin{aligned}\hat{\omega}_H : \mathscr{H}_{\mathcal{C}'} &\to \mathscr{H}_H, \\ \phi^{\mathcal{C}'} &\mapsto \hat{\omega}_H(\phi) = \phi^H = (\omega_H)^{-1}\phi^{\mathcal{C}'}.\end{aligned}$$

In particular one can introduce:

$$\begin{aligned}\hat{\Xi}_H : \mathscr{H}_0 &\to \mathscr{H}_H, \\ \phi^0 &\mapsto \hat{\Xi}_H(\phi^0) = \hat{\omega}_H(\hat{\omega}_0^{-1}(\phi^0)) = \phi^H,\end{aligned}$$

linking the Hilbert space of minkowskian solutions to the desitterian ones.

Quantizing the scalar field

The rescaling as an unitary map (2/2)

- Then, an endomorphism on, say, $\mathscr{H}_0$ is carried to an end. of $\mathscr{H}_H$ by the usual formula: $\mathcal{O}_H = \hat{\Xi}_H \mathcal{O}_0 \hat{\Xi}_H^{-1}$.
- Thanks to Riesz lemma the same holds for linear functionals.
- Each Hilbert space $\mathscr{H}_{\mathcal{C}'}$, $\mathscr{H}_{H\geq 0}$ carries the same uir of $SO_0(2,4)$ and the map $\hat{\omega}_H$ is an unitary intertwining operator between those different realizations.

Quantizing the scalar field

The two point function (1/2)

Now, everything is settled on dS/Mink/$\mathcal{C}'$ and out of the modes one constructs the autoreproducible kernel/two point function:

$$D^+_H(y, y') = \sum_{Llm} \phi_{Llm}(y)\phi^*_{Llm}(y') = \frac{-H^2}{8\pi^2}\frac{1}{\mathcal{Z}_\epsilon - 1},$$

with $\mathcal{Z}$ a quantity defined on the whole dS manifold and ϵ a regulator.

When y and y' can be linked by a geodesic of length μ, one has:

$$\mathcal{Z} = \begin{cases} \cosh(H\mu) & \text{for timelike separated points,} \\ 1 & \text{for lightlike separated points,} \\ \cos(H\mu) & \text{for spacelike separated points.} \end{cases}$$

Quantizing the scalar field

The two point function (2/2)

Then, one finds that the two point function fulfills all the required properties: $SO_0(1,4)$ covariance, positive conformal energy, hermiticity, local commutation, positivity.

In fact the covariance is extended from $SO_0(1,4)$ to $SO_0(2,4)$.

When the points y and y' are geodesically linked one has:

$$D_H^+(y,y') = \frac{-1}{8\pi^2}\Big(\frac{1}{\sigma_\epsilon} - \frac{H^2}{6} + \frac{H^4}{60}\sigma - \frac{H^6}{756}\sigma^2 + \ldots\Big),$$

with $\sigma = \mu^2/2$.

The vacuum exhibits the Hadamard behavior, is “Bunch-Davies”, “Euclidian”, “Conformal”, analytic in the extended forward tube, geodesic-KMS at $\beta = 2\pi H$, ... in short *the good one*.

Quantizing the Maxwell field

Maxwell's equations

Written in term of the potential A they are:

$$\Box A^\mu - \nabla_\nu \nabla^\mu A^\nu = 0,$$

which on de Sitter spacetime read:

$$\left(\Box + 3H^2\right) A^\mu - \nabla^\mu \nabla \cdot A = 0.$$

They feature two properties:

- gauge invariance, that is: $A^\mu = \nabla^\mu \phi$ is a solution $\forall \phi$,
- conformal invariance.

Idea. Overcome the former while using/preserving the latter.
Rmk. From now on no mention of Mink. is made, it works as previously exposed.

Quantizing the Maxwell field

Gauge invariance → obstruction to quantization

Gauge invariance might be taken as the fact that:

- there exists unconstrained (trivial) solutions,
- there is no non-trivial two point function (manifestly covariant, fulfilling Maxwell's eq., ...) [Morchio, Strocchi],
- is not a well-posed Cauchy problem,
- there exists solutions which are, wrt *a certain scalar product*, orthogonal to all solutions including themselves (pure gauge states),
- ...

in short troubles ahead.

Quantizing the Maxwell field

The general idea of a Gupta-Bleuler quantization

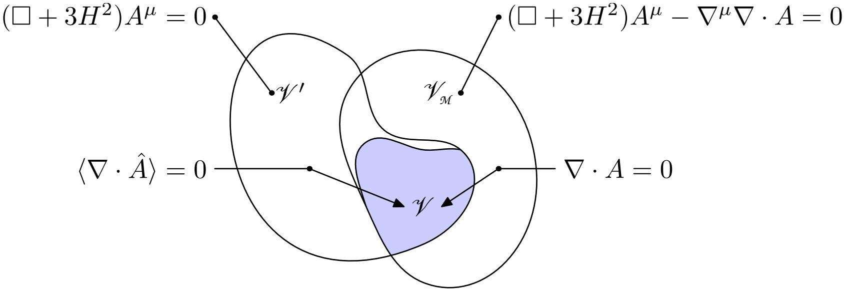
The scheme follows these steps:

1. start with a space of solutions $\mathscr{V}_{\mathscr{M}}$ which is gauge invariant → one can *not* quantize canonically the field (this is the source of our problems),
2. specify a *"physical"* subspace $\mathscr{V}$ through a gauge fixing equation $G(A) = 0$,
3. use an auxiliary space $\mathscr{V}'$ on which, by hypothesis, one can quantize the field $A \mapsto \hat{A}$,
4. select the Fock subspace s.t. $\langle \phi, G'(\hat{A})\psi \rangle = 0$ holds and reduce to $\mathscr{F}(\mathscr{V})$.

Rmk. The auxiliary space $\mathscr{V}'$ has to be chosen s.t. there exists solutions which overlap with pure gauge solutions (which are not orthogonal to pure gauge states, leads to states of negative norm and a quantization in Krein space).

Quantizing the Maxwell field

(The Lorenz gauge)



$(\Box + 3H^2)A^\mu = 0$

$(\Box + 3H^2)A^\mu - \nabla^\mu \nabla \cdot A = 0$

$\mathscr{V}'$

$\mathscr{V}_{\mathscr{M}}$

$\langle \nabla \cdot \hat{A} \rangle = 0$

$\mathscr{V}$

$\nabla \cdot A = 0$

Rmk. Except the space $\mathscr{V}_{\mathscr{M}}$ none of these spaces is invariant under the action of $SO_0(2,4)$.
Goal. Keep the conformal, i.e. $SO_0(2,4)$, invariance for each of these spaces, i.e. for the gauge fixing equation, etc.

Quantizing the Maxwell field

The Eastwood-Singer gauge condition $G(A) = 0$

Recall. We wish to keep the conformal invariance.

The gauge condition of Eastwood-Singer:

$$G(A) = \left[\Box \nabla \cdot A + 2\left(\nabla^{\mu} R_{\mu\nu} A^{\nu}\right) - \frac{2}{3}\left(\nabla_{\mu} R A^{\mu}\right) \right] = 0$$

is conformally invariant *on the space of solutions* of Maxwell's equations.

Rmk. Setting $A_{\mu} = \nabla_{\mu}\phi$ one gets the Paneitz operator.

On the de Sitter spacetime this *physical* subspace reads:

$$\mathscr{V} = \{A \in \mathscr{V}_{\mathscr{M}} \mid G(A) = (\Box + 2H^2)\nabla \cdot A = 0\} \subset \mathscr{V}_{\mathscr{M}}$$

and is left invariant under the action of $SO_0(2,4)$.

M.G. Eastwood and M. Singer, A conformally invariant Maxwell gauge, Phys. Lett., A107:7374, (1985).

Quantizing the Maxwell field

The auxiliary space $\mathscr{V}'$

To produce an auxiliary space $\mathscr{V}'$ which is kept invariant under $SO_0(2,4)$ we return in $\mathbb{R}^6$ and consider the, previously used, equations:

$$\left(\Box_6 a^\alpha\right)|_{\mathcal{C}} = 0, \qquad\qquad z \cdot \frac{\partial}{\partial z} a^\alpha = -a^\alpha .$$

But, this is insufficient and some additional work needs to be done.

Quantizing the Maxwell field

Decomposition of the vectorial field of $\mathbb{R}^6$ to one vector and two scalars

The vectorial field a, homogeneous of its variables, can be broken into:

$$\begin{aligned} a &= a^{\alpha}(z)\frac{\partial}{\partial z^{\alpha}} \\ &= hA^{h}\frac{\partial}{\partial h} + \left(\overline{A}^{\kappa} + \frac{1}{y^2}y^{\kappa}\psi\right)\frac{\partial}{\partial y^{\kappa}} \end{aligned}$$

such that:

- A^h, ψ are scalar fields under $SO_0(1,4)$,
- $\overline{A}^{\kappa}$ is a vectorial field under $SO_0(1,4)$.

Rmk. The variables $\{y^{\kappa}\}$ can be recognized as the ambient space approach in $\mathbb{R}^5$ of de Sitter fields.

Quantizing the Maxwell field

Indecomposable action of $SO_0(2,4)$ on the triplet of fields

The decomposition of a^α is done in a "clever" way s.t. schematically:

$$\begin{pmatrix} A'^h \\ \overline{A}'^\kappa \\ \psi' \end{pmatrix} = \begin{pmatrix} T & & \\ & T & \\ & & T \end{pmatrix} \begin{pmatrix} A^h \\ \overline{A}^\kappa \\ \psi \end{pmatrix}, \quad \begin{pmatrix} A'^h \\ \overline{A}'^\kappa \\ \psi' \end{pmatrix} = \begin{pmatrix} T & T & T \\ & T & T \\ & & T \end{pmatrix} \begin{pmatrix} A^h \\ \overline{A}^\kappa \\ \psi \end{pmatrix},$$

$$g \in SO_0(1,4) \subset SO_0(2,4), \qquad g \in SO_0(2,4) \setminus SO_0(1,4),$$

That is, $SO_0(2,4)$ has an indecomposable action on the triplet of fields with the following hierarchy:

$$\psi \longrightarrow \overline{A}^\kappa \longrightarrow A^h.$$

Thus, the equation: $\psi = 0$ is conformally invariant.

Quantizing the Maxwell field

The field equation projected onto the de Sitter spacetime

The equation $\Box_6 a = 0$ induces on the triplet of desitterian fields:

$$\begin{aligned}
&\big(\Box + 2H^2\big)A^h = 0,\\
&\big(\Box + 3H^2\big)A^\mu + 2\nabla^\mu(A^h - H^2\psi) = 0,\\
&\big(\Box + 6H^2\big)\psi - 4A^h - 2\nabla\cdot A = 0.
\end{aligned}$$

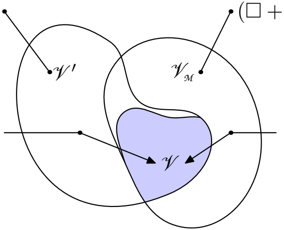
Rmk. for $\psi = 0$ one recovers Maxwell's equations <u>plus</u> the Eastwood-Singer gauge fixing equation.

Rmk. On the space of solutions $\mathscr{V}'$ with $\psi = 0$ the indecomposable action of $SO_0(2,4)$ coincides with the natural one.

Rmk. These additional fields might be thought of as Nakanishi-Lautrup-like auxiliary fields.

Quantizing the Maxwell field

Summarizing the conformally invariant Gupta-Bleuler scheme



$$(\Box + 2H^2)A^h = 0$$

$$(\Box + 3H^2)A^\mu + 2\nabla^\mu(A^h - H^2\psi) = 0$$

$$(\Box - 2H^2)\psi - 2\nabla \cdot A - 4A^h = 0$$

$$(\Box + 3H^2)A^\mu - \nabla^\mu\nabla \cdot A = 0$$

$$\langle\hat{\psi}\rangle = 0$$

$$(\Box + 2H^2)\nabla \cdot A = 0$$

Quantizing the Maxwell field

Solving the equations

Now all is left to do is to solve:

$$\left(\Box_6 a^\alpha\right)|_{\mathcal{C}} = 0, \qquad\qquad z \cdot \frac{\partial}{\partial z} a^\alpha = -a^\alpha .$$

which is done thanks to the scalar solutions.

Then, one equips these solutions with the scalar product:

$$\langle a, b \rangle = -i\eta_{\alpha\beta} \int_{\beta=0} ((a^\alpha)^* \overleftrightarrow{\partial_\beta} b^\beta)\mathrm{d}\mathrm{Vol}(S^3)\Big|_{\mathcal{C}'} .$$

- The representation of $SO_0(2,4)$ is unitary wrt to the sp.
- The physical subspace $\mathscr{V}$ is proven to be of positive norm.

Quantizing the Maxwell field

A side remark on the scalar product

The scalar product between solutions induced by the one in $\mathbb{R}^6$ in terms of dS quantities reads, reduced at $\psi = 0$ and using the eom,:

$$\langle A_1, A_2 \rangle = -i \int [(A_1)^*_\mu \overleftrightarrow{\nabla}_0 A_2^\mu - ((A_1^*)_0 \nabla \cdot A_2 - (A_2)_0 \nabla \cdot A_1^*)] d\Sigma^0$$

which is <u>not</u> the Klein-Gordon product associated to $(\Box_H + 3H^2) A^\mu = 0$.

One checks that, as mentioned earlier, the pure gauge solutions are indeed orthogonal to any solutions (including themselves) of Maxwell's equations.

Of course the conformal invariance is kept, in general.

Quantizing the Maxwell field

The projected two-point functions on de Sitter spacetime

The modes can be summed st. the kernel is autoreproducible and invariant under $SO_0(2,4)$. Projected on dS it yields:

$$D^{\psi\psi}(y,y') = \frac{-1}{8\pi^2}, \qquad D^{\psi\vartheta}(y,y') = \frac{H^2}{8\pi^2}\frac{\Theta'^{\vartheta}\cdot y}{\mathcal{Z}_\epsilon - 1},$$

$$D^{\kappa\vartheta}(y,y') = \frac{H^2}{8\pi^2}\frac{\Theta^{\kappa}\cdot\Theta'^{\vartheta}}{\mathcal{Z}_\epsilon - 1}, \qquad D^{\psi h}(y,y') = \frac{H^2}{8\pi^2}\frac{1}{\mathcal{Z}_\epsilon - 1},$$

$$D^{\kappa h}(y,y') = 0, \qquad D^{hh}(y,y') = \frac{H^4}{8\pi^2}\frac{1}{\mathcal{Z}_\epsilon - 1},$$

in the ambient space formalism of dS fields in $\mathbb{R}^5$.

Rmk. No *Lorenz* gauge-like quantization can produce that two-point function.
Rmk. This is the minimal form of a vectorial two-point function on dS.

Quantizing the Maxwell field

Translation in the intrinsic formalism

On the physical subspace $\mathscr{F}(\mathscr{V})$ only the vectorial two point function matters.

Given two points p and p' linked by a geodesic one introduces:

- $g^{\mu}{}_{\nu'}(x,x')$ the parallel propagator ($\Leftrightarrow$ parallel transport of a vector along the given geodesic),
- $n^{\mu}(x,x')$ the outgoing unit tangent vector at T_pM,
- $n^{\nu'}(x,x')$ the outgoing unit tangent vector at $T_{p'}M$.

In a maximally symmetric spacetime they constitute a basis of bitensors and any covariant two point function reads:

$$D_{\mu\nu'}(x,x') = \alpha(\mathcal{Z})g_{\mu\nu'}(x,x') + \beta(\mathcal{Z})n_{\mu}(x,x')n_{\nu'}(x,x').$$

Quantizing the Maxwell field

The two point function

In the usual setting (Lorenz gauge) Allen and Jacobson find:

$$D^{\mathrm{AJ}}_{\mu\nu'}(x,x') = \alpha(\mathcal{Z}) g_{\mu\nu'}(x,x') + \beta(\mathcal{Z}) n_\mu(x,x') n_{\nu'}(x,x'),$$

where:

$$\alpha(\mathcal{Z}) = \frac{H^2}{24\pi^2}\left[-\frac{3}{\mathcal{Z}-1} + \frac{1}{\mathcal{Z}+1} + \left(\frac{2}{\mathcal{Z}+1} + \frac{2}{(\mathcal{Z}+1)^2}\right)\log\left(\frac{\mathcal{Z}-1}{2}\right)\right],$$

$$\beta(\mathcal{Z}) = \frac{H^2}{24\pi^2}\left[1 - \frac{2}{\mathcal{Z}+1} + \left(\frac{2}{\mathcal{Z}+1} + \frac{4}{(\mathcal{Z}+1)^2}\right)\log\left(\frac{\mathcal{Z}-1}{2}\right)\right].$$

While using the Eastwood-Singer gauge in our quantization yields:

$$\boxed{D_{\mu\nu'}(x,x') = \frac{H^2}{8\pi^2}\left(\frac{g_{\mu\nu'}}{\mathcal{Z}_\epsilon - 1} - n_\mu n_{\nu'}\right).}$$

Allen, B. Jacobson, T., Vector two point functions in maximally symmetric spaces, *Commun. Math. Phys.* **103**, *(1986)*.

Conclusion

- By keeping the conformal invariance we do find a simple, compact and manifestly covariant two point function for the quantization of the Maxwell field.
- It carries the same physics as the result of Allen and Jacobson since the two point function of the $F_{\mu\nu}$, that is:$\nabla^{[\mu}\nabla_{[\sigma'}D^{\nu]}{}_{\rho']}$, which is gauge invariant, is exactly the same.
- Moreover, by construction, our solutions and then the two point functions have trivial null curvature limit.
- At each step in our quantization process the action of both $SO_0(2,4)$ and $SO_0(1,4)$ remains clear.

Outlooks

- Can it be extended to $[\eta]$? (yes)
- The geometric apparatus can be used to encompass massive scalar fields and some cases of external potentials on dS, while spoiling the $SO_0(2,4)$ invariance, thanks to appropriate potentials in $\mathbb{R}^6$ (thanks to N. Pinamonti).
- The Eastwood-Singer equation can be (sort of) generalized ...
- Finally consider the interacting qed on dS. Previous calculations [Bros, Epstein, Moschella] have already shown that new process (for scalar fields) appear as soon as dS is involved. Would that be the case in a qed?