A geometric interpretation of the sine-Gordon equation

We write the sine-Gordon equation as

$$\phi_{tt} - c^2 \phi_{xx} = \frac{1}{\rho^2} \sin\phi$$

$$\Leftrightarrow \quad \phi_{uv} = \frac{1}{\rho^2} \sin\phi \quad \text{with the light-cone (chiral) coordinates: } \left\| \begin{array}{l} u = t + c^{-1}x \\ v = t - c^{-1}x \end{array} \right.$$

We would like here to show how solutions of this equation are related to some surfaces embedded in $\mathbb{R}^3$.

Let us look, to begin, at the motion through time of a curve C. To any point of a (differentiable) curve we can attach an orthonormalized triad $\{\vec{v}, \vec{n}, \vec{b}\}$ with:

$\vec{v}$ the tangent to the curve C
$\vec{n}$ — normal ——
$\vec{b}$ — binormal ——, $\vec{b} = \vec{v} \times \vec{n}$

Let us call l the "natural length" on the curve, we have then the Serret-Frenet system

$$\left\| \begin{array}{l} \vec{v}_l = k\vec{n} \\ \vec{n}_l = -k\vec{n} + \tau\vec{b} \\ \vec{b}_l = -\tau\vec{n} \end{array} \right.$$

with k the curvature of C, τ its torsion, at the point P.

Let us now consider a time evolution s.t. the triad $\{\vec{v}, \vec{n}, \vec{b}\}$ remains orthonormalized through time

From $\|\vec{v}\|^2 = 1$ one gets $\dot{\vec{v}}.\vec{v} = 0 \Leftrightarrow \dot{\vec{v}} = a\vec{n} + b\vec{b}$
and similarly

$$\dot{\vec{n}} = c\vec{v} + d\vec{b}$$
$$\dot{\vec{b}} = e\vec{v} + f\vec{n}$$

In addition $\vec{v}.\vec{n} = 0 \Rightarrow \dot{\vec{v}}.\vec{n} + \vec{v}.\dot{\vec{n}} = 0$
$\Rightarrow a + c = 0$
and similarly $\vec{v}.\vec{b} = 0 \Rightarrow b + e = 0$
$\vec{n}.\vec{b} = 0 \Rightarrow d + f = 0$

Set $a = \alpha$, $b = \beta$, $d = \gamma$ and one gets

$$\boxed{\begin{array}{l} \vec{v}_t = \alpha\vec{n} + \beta\vec{b} \\ \vec{n}_t = -\alpha\vec{v} + \gamma\vec{b} \\ \vec{b}_t = -\beta\vec{v} - \gamma\vec{n} \end{array}} \quad \text{or} \quad \begin{pmatrix} \vec{v}_t \\ \vec{n}_t \\ \vec{b}_t \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{n} \\ \vec{b} \end{pmatrix}$$

Remark. Attention! The $k, \tau, \alpha, \beta, \gamma$ are (to this point) functions of t and l.

We now ask the compatibility conditions ($\Leftrightarrow$ inextensible curve) (actually integrability cond.)

$$\boxed{\vec{v}_{tl} = \vec{v}_{lt} \; ; \; \vec{n}_{tl} = \vec{n}_{lt} \; ; \; \vec{b}_{tl} = \vec{b}_{lt}}$$

for the first one:

$$\begin{aligned} \vec{v}_{tl} = (\vec{v}_t)_l = (\alpha\vec{n} + \beta\vec{b})_l &= \alpha_l\vec{n} + \alpha\vec{n}_l + \beta_l\vec{b} + \beta\vec{b}_l \\ &= \alpha_l\vec{n} - k\alpha\vec{v} + \tau\alpha\vec{b} + \beta_l\vec{b} - \tau\beta\vec{n} \end{aligned}$$

$$= -k\alpha\vec{v} + (\alpha_\ell - \tau\beta)\vec{n} + (\tau\alpha + \beta_\ell)\vec{b}$$

and $\vec{v}_{\ell t} = (\vec{v}_\ell)_t = (k\vec{n})_t = k_t\vec{n} + k\vec{n}_t = k_t\vec{n} - \alpha k\vec{v} + k\gamma\vec{b}$

$$= -k\alpha\vec{v} + k_t\vec{n} + k\gamma\vec{b}$$

equating $\vec{v}_{t\ell}$ and $\vec{v}_{\ell t}$ yields :

$$-k\alpha = -k\alpha \quad \text{(tautology)}$$
$$\alpha_\ell - \tau\beta = k_t$$
$$\beta_\ell + \tau\alpha = k\gamma$$

for the second one gets:

$$\vec{n}_{t\ell} = (\vec{n}_t)_\ell = (-\alpha\vec{v} + \gamma\vec{b})_\ell = -\alpha_\ell\vec{v} - \alpha\vec{v}_\ell + \gamma_\ell\vec{b} + \gamma\vec{b}_\ell$$
$$= -\alpha_\ell\vec{v} - \alpha k\vec{n} + \gamma_\ell\vec{b} - \gamma\tau\vec{n}$$
$$= -\alpha_\ell\vec{v} - (\alpha k + \gamma\tau)\vec{n} + \gamma_\ell\vec{b}$$

$$\vec{n}_{\ell t} = (\vec{n}_\ell)_t = (-k\vec{v} + \tau\vec{b})_t = -k_t\vec{v} - k\vec{v}_t + \tau_t\vec{b} + \tau\vec{b}_t$$
$$= -k_t\vec{v} - k\alpha\vec{n} - k\beta\vec{b} + \tau_t\vec{b} - \tau\beta\vec{v} - \tau\gamma\vec{n}$$
$$= -(k_t + \tau\beta)\vec{v} - (k\alpha + \tau\gamma)\vec{n} + (\tau_t - k\beta)\vec{b}$$

equating both yields :

$$\alpha_\ell = k_t + \tau\beta \quad \text{(already recorded)}$$
$$k\alpha + \gamma\tau = k\alpha + \tau\gamma \quad \text{(tautology)}$$
$$\gamma_\ell = \tau_t - k\beta$$

For the third and final

$$\vec{b}_{t\ell} = (\vec{b}_t)_\ell = (-\beta\vec{v} - \gamma\vec{n})_\ell = -\beta_\ell\vec{v} - \beta\vec{v}_\ell - \gamma_\ell\vec{n} - \gamma\vec{n}_\ell$$
$$= -\beta_\ell\vec{v} - \beta k\vec{n} - \gamma_\ell\vec{n} + \gamma k\vec{v} - \gamma\tau\vec{b}$$
$$= (\gamma k - \beta_\ell)\vec{v} - (\beta k + \gamma_\ell)\vec{n} - \gamma\tau\vec{b}$$

$$\vec{b}_{\ell t} = (\vec{b}_\ell)_t = (-\tau\vec{n})_t = -\tau_t\vec{n} - \tau\vec{n}_t$$
$$= -\tau_t\vec{n} + \alpha\tau\vec{v} - \gamma\tau\vec{b}$$
$$= \alpha\tau\vec{v} - \tau_t\vec{n} - \gamma\tau\vec{b}$$

Which gives :

$$\alpha\tau = \gamma k - \beta_\ell \quad \text{(already recorded)}$$
$$\beta k + \gamma_\ell = \tau_t \quad \text{(already recorded)}$$
$$\gamma\tau = \gamma\tau \quad \text{(tautology)}$$

The compatibility conditions thus yields

$$\boxed{\begin{aligned} \alpha_\ell &= k_t + \tau\beta \\ \beta_\ell &= k\gamma - \tau\alpha \\ \gamma_\ell &= \tau_t - k\beta \end{aligned}}$$

A simple consequence of the system can be worked out:
multiply the first eq. by α,
——— 2nd ——— β,
——— 3rd ——— γ and take the sum

$$\alpha\alpha_\ell + \beta\beta_\ell + \gamma\gamma_\ell = \alpha k_t + \cancel{\alpha\tau\beta} + \cancel{\beta k\gamma} - \cancel{\beta\tau\alpha} + \gamma\tau_t - \cancel{\gamma k\beta}$$

$$\rightarrow \quad \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)_\ell = \alpha k_t + \gamma\tau_t$$

If the right hand side is zero the equations are greatly simplified, this happens for

- $(\alpha, \gamma) = 0$
- $(k_t, \tau_t) = 0$
- $(\alpha, \tau_t) = 0$
- $(k_t, \gamma) = 0$

The first one can't be considered as it leads to a too simple motion through time.
The second can't be considered too as it leads to a too simple curve (straight line or so).

Let us consider the third case and set

$$\left| \begin{aligned} \beta(t,\ell) &= \delta(t)\sin\phi(t,\ell) \\ \gamma(t,\ell) &= \delta(t)\cos\phi(t,\ell) \end{aligned} \right.$$

δ function of t only since $(\beta^2+\gamma^2)_\ell = (\delta^2)_\ell = 0$

If we plug that into the system one gets

$$\begin{aligned} 0 &= k_t + \tau\delta \sin\phi \\ \delta\cos(\phi)\,\phi_\ell &= k\delta\cos\phi \\ -\delta\sin(\phi)\,\phi_\ell &= -k\delta\sin\phi \end{aligned}$$

The 2nd and 3rd equation give $k = \phi_\ell$ plugged into the 1st yields:

$$\phi_{t\ell} = -\tau\delta\sin\phi$$

$\tau \equiv \tau(\ell)$. $\delta \equiv \delta(t)$ if one choose τ and δ as constant one can turn the above equation into

$$\boxed{\phi_{t\ell} = \frac{1}{\rho^2}\sin(\phi)}$$

Which is the sine-Gordon equation we seeked.

$$\begin{pmatrix} \vec{v} \\ \vec{n} \\ \vec{b} \end{pmatrix}_\ell = \begin{pmatrix} 0 & \phi_\ell(t,\ell) & 0 \\ -\phi_\ell(t,\ell) & 0 & -1/\rho \\ 0 & -1/\rho & 0 \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{n} \\ \vec{b} \end{pmatrix}$$

$$\begin{pmatrix} \vec{v} \\ \vec{m} \\ \vec{b} \end{pmatrix}_t = \begin{pmatrix} 0 & 0 & \frac{1}{\rho}\sin\phi(t,\ell) \\ 0 & 0 & \frac{1}{\rho}\cos\phi(t,\ell) \\ -\frac{1}{\rho}\sin\phi(t,\ell) & -\frac{1}{\rho}\cos\phi(t,\ell) & 0 \end{pmatrix} \begin{pmatrix} \vec{v} \\ \vec{m} \\ \vec{b} \end{pmatrix}$$

Notice that for $(k_t, \gamma) = 0$ one gets too the sine-Gordon equation.

Now we have seen that the motion through time of a certain type of curves (generates) entails that the "angle" $\phi(t,\ell)$ has to fulfill the sine-Gordon equation.

Motion through time of a curve $\Leftrightarrow$ embedded surface in $\mathbb{R}^3$

Here these constant torsion curve describe pseudo-spherical surface in $\mathbb{R}^3$

pseudo-spherical $\equiv$ constant Gaussian curvature

$$K = \frac{\det \mathrm{I}}{\det \mathrm{II}} = \frac{-1}{\rho^2}$$

with $\vec{r}(u,v)$ the vector describing the surface Σ

$\vec{r}_u = \partial_u \vec{r}$, $\vec{r}_v = \partial_v \vec{r}$ the tangent vectors to the surface,

$\vec{N} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$ the normalized normal vector.

and $\mathrm{I} = d\vec{r}.d\vec{r}$ the first form of $\Sigma \subset \mathbb{R}^3$

$\mathrm{II} = -d\vec{N}.d\vec{r}$ — 2nd —

Now suppose that we have a mapping of one such surface Σ onto another surface Σ' of this type. We would then have a (induced) map of a solution of sine-Gordon eq. to another solution of sine-Gordon eq.

One can find one such mapping, Bäcklund found

$B_\alpha(\phi) = \phi^\alpha$, ϕ = "old" solution (seed)

ϕ^α = "new" solution

$$B_\alpha : \boxed{\begin{aligned} \phi^\alpha_t &= \phi_t + \frac{2\alpha}{\rho}\sin\left(\frac{\phi^\alpha+\phi}{2}\right) \\ \phi^\alpha_\ell &= -\phi_\ell + \frac{2}{\alpha\rho}\sin\left(\frac{\phi^\alpha-\phi}{2}\right) \end{aligned}}$$

This is a PDE system which has to be integrated.

First let us verify that ϕ^α is actually a solution provided that ϕ is already a solution

$$\partial_l \phi^\alpha_t = \phi_{lt} + \frac{\cancel{2}\alpha}{\rho}\cos\left(\frac{\phi^\alpha+\phi}{2}\right)\times\left(\frac{\phi^\alpha_l+\phi_l}{\cancel{2}}\right)$$

$$= \phi_{lt} + \frac{\alpha}{\rho}\cos\left(\frac{\phi^\alpha+\phi}{2}\right)\times\left(-\phi_l + \frac{2}{\alpha\rho}\sin\left(\frac{\phi^\alpha-\phi}{2}\right) + \phi_l\right)$$

$$= \phi_{lt} + \frac{2}{\rho^2}\cos\left(\frac{\phi^\alpha+\phi}{2}\right)\sin\left(\frac{\phi^\alpha-\phi}{2}\right)$$

$$= \phi_{lt} + \frac{2}{\rho^2}\left[\left(\cos\frac{\phi^\alpha}{2}\cos\frac{\phi}{2} - \sin\frac{\phi^\alpha}{2}\sin\frac{\phi}{2}\right)\times\left(\sin\frac{\phi^\alpha}{2}\cos\frac{\phi}{2} - \cos\frac{\phi^\alpha}{2}\sin\frac{\phi}{2}\right)\right]$$

$$= \phi_{lt} + \frac{2}{\rho^2}\left[\cos\frac{\phi^\alpha}{2}\sin\frac{\phi^\alpha}{2}\cos^2\frac{\phi}{2} - \cos^2\frac{\phi^\alpha}{2}\cos\frac{\phi}{2}\sin\frac{\phi}{2} - \sin^2\frac{\phi^\alpha}{2}\cos\frac{\phi}{2}\sin\frac{\phi}{2} + \cos\frac{\phi^\alpha}{2}\sin\frac{\phi^\alpha}{2}\sin^2\frac{\phi}{2}\right]$$

$$= \phi_{lt} + \frac{2}{\rho^2}\left(\cos\frac{\phi^\alpha}{2}\sin\frac{\phi^\alpha}{2} - \cos\frac{\phi}{2}\sin\frac{\phi}{2}\right)$$

$$= \phi_{lt} + \frac{1}{\rho^2}\left(\sin\phi^\alpha - \sin\phi\right)$$

$$= \frac{1}{\rho^2}\sin\phi^\alpha$$

So, yes, provided ϕ is a solution ϕ^α is also a solution $\forall\alpha$

$\Rightarrow$ Let us use that. An obvious solution is $\phi=0$

$$B_\alpha(0):\quad \phi^\alpha_t = \frac{2\alpha}{\rho}\sin\left(\frac{\phi^\alpha}{2}\right)$$

$$\phi^\alpha_l = \frac{2}{\alpha\rho}\sin\left(\frac{\phi^\alpha}{2}\right) = \frac{1}{\alpha^2}\times\frac{2\alpha}{\rho}\sin\left(\frac{\phi^\alpha}{2}\right) = \frac{1}{\alpha^2}\phi^\alpha_t$$

This link between the derivatives actually implies

$$\phi^\alpha(t,l) = \phi^\alpha\left(t+\frac{1}{\alpha^2}l\right) = \phi^\alpha(u)$$

We then have $\partial_u\phi^\alpha = \frac{2\alpha}{\rho}\sin\left(\frac{\phi^\alpha}{2}\right)$

$$\Rightarrow \int_{\phi^\alpha_0}^{\phi^\alpha}\frac{d\phi^\alpha}{\sin\left(\frac{\phi^\alpha}{2}\right)} = \frac{2\alpha}{\rho}(u-u_0)$$

as per usual we consider the half-angle

$$\sin\left(\frac{\phi^\alpha}{2}\right) = 2\cos\left(\frac{\phi^\alpha}{4}\right)\sin\left(\frac{\phi^\alpha}{4}\right) = 2\cos^2\left(\frac{\phi^\alpha}{4}\right)tg\left(\frac{\phi^\alpha}{4}\right)$$

$$= \frac{2\,tg(\phi^\alpha/4)}{1+tg^2(\phi^\alpha/4)} = \frac{2t}{1+t^2}$$

with $t = tg\left(\frac{\phi^\alpha}{4}\right)$; $dt = (1+t^2)\frac{1}{4}d\phi^\alpha$

then $$\int\frac{d\phi^\alpha}{\sin\left(\frac{\phi^\alpha}{2}\right)} = \int\frac{1+t^2}{2t}\times\frac{4}{1+t^2}dt = 2\int\frac{dt}{t} = 2\ln\left(\frac{t}{t_0}\right)$$

$$\Rightarrow \ln\left(\frac{t}{t_0}\right) = \frac{\alpha}{\rho}(u-u_0) \Rightarrow t = t_0\,e^{\frac{\alpha}{\rho}(u-u_0)}$$

$$\Rightarrow \phi^\alpha = 4\,\mathrm{arctg}\left(t_0\,e^{\frac{\alpha}{\rho}(u-u_0)}\right)$$

$$\boxed{\phi^\alpha = 4\,\mathrm{arctg}\left(t_0\,e^{\frac{\alpha}{\rho}\left(t+\frac{1}{\alpha^2}l-u_0\right)}\right)}$$

with α, u_0, t_0 arbitrary real constants

ϕ^α is the 1-soliton solution.

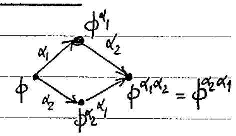
We can iterate the Bäcklund transformation $B_{\alpha_2}(\phi^{\alpha_1}) = \phi^{\alpha_1\alpha_2}, \ldots$ and this could be terribly complicated (remember system of coupled PDE).

A nice, and remarkable, way to get around that is to ask the commutativity of Bäcklund transf.

$$B_{\alpha_2}(B_{\alpha_1}(\phi)) = B_{\alpha_1}(B_{\alpha_2}(\phi)) , \quad \phi^{\alpha_1\alpha_2} = \phi^{\alpha_2\alpha_1}$$

This is the content of Bianchi' <u>permutability theorem</u>

Let us write the result



$$\phi_t^{\alpha_1\alpha_2} = \phi_t^{\alpha_1} + \frac{2\alpha_2}{\rho}\sin\left(\frac{\phi^{\alpha_1\alpha_2}+\phi^{\alpha_1}}{2}\right) \quad (A)$$

$$\phi_t^{\alpha_1} = \phi_t + \frac{2\alpha_1}{\rho}\sin\left(\frac{\phi^{\alpha_1}+\phi}{2}\right) \quad (B)$$

$$\phi_t^{\alpha_2\alpha_1} = \phi_t^{\alpha_2} + \frac{2\alpha_1}{\rho}\sin\left(\frac{\phi^{\alpha_2\alpha_1}+\phi^{\alpha_2}}{2}\right) \quad (C)$$

$$\phi_t^{\alpha_2} = \phi_t + \frac{2\alpha_2}{\rho}\sin\left(\frac{\phi^{\alpha_2}+\phi}{2}\right) \quad (D)$$

ask that $\phi^{\alpha_1\alpha_2} = \phi^{\alpha_2\alpha_1}$ and consider $A+B-C-D$:

$$\Rightarrow \phi^{\alpha_1} - \phi^{\alpha_2} = \phi^{\alpha_1} - \phi^{\alpha_2} + \frac{2\alpha_2}{\rho}\sin\left(\frac{\phi^{\alpha_1\alpha_2}+\phi^{\alpha_1}}{2}\right) + \frac{2\alpha_1}{\rho}\sin\left(\frac{\phi^{\alpha_1}+\phi}{2}\right)$$
$$- \frac{2\alpha_1}{\rho}\sin\left(\frac{\phi^{\alpha_2\alpha_1}+\phi^{\alpha_2}}{2}\right) - \frac{2\alpha_2}{\rho}\sin\left(\frac{\phi^{\alpha_2}+\phi}{2}\right)$$

$$\Rightarrow \alpha_1\left[\sin\left(\frac{\phi^{\alpha_2\alpha_1}+\phi^{\alpha_2}}{2}\right) - \sin\left(\frac{\phi^{\alpha_1}+\phi}{2}\right)\right]$$
$$= \alpha_2\left[\sin\left(\frac{\phi^{\alpha_1\alpha_2}+\phi^{\alpha_1}}{2}\right) - \sin\left(\frac{\phi^{\alpha_2}+\phi}{2}\right)\right]$$

We use the formulas on the difference of two sinuses.

$$\cancel{2\alpha_1\cos\left(\frac{\phi^{\alpha_2\alpha_1}+\phi^{\alpha_2}+\phi^{\alpha_1}+\phi}{4}\right)}\sin\left(\frac{\phi^{\alpha_2\alpha_1}+\phi^{\alpha_2}-\phi^{\alpha_1}-\phi}{4}\right)$$
$$= \cancel{2\alpha_2\cos\left(\frac{\phi^{\alpha_1\alpha_2}+\phi^{\alpha_1}+\phi^{\alpha_2}+\phi}{4}\right)}\sin\left(\frac{\phi^{\alpha_1\alpha_2}+\phi^{\alpha_1}-\phi^{\alpha_2}-\phi}{4}\right)$$

$$\Rightarrow \alpha_1\sin\left(\frac{\phi^{\alpha_2\alpha_1}-\phi+\phi^{\alpha_2}-\phi^{\alpha_1}}{4}\right) = \alpha_2\sin\left(\frac{\phi^{\alpha_2\alpha_1}-\phi-(\phi^{\alpha_2}-\phi^{\alpha_1})}{4}\right)$$

$$\Rightarrow \alpha_1\left(\cos\left(\frac{\phi^{\alpha_2\alpha_1}-\phi}{4}\right)\sin\left(\frac{\phi^{\alpha_2}-\phi^{\alpha_1}}{2}\right) + \sin\left(\frac{\phi^{\alpha_2\alpha_1}-\phi}{4}\right)\cos\left(\frac{\phi^{\alpha_2}-\phi^{\alpha_1}}{4}\right)\right)$$

$$= \alpha_2\left(\text{———}\,''\,\text{———} - \text{———}\,''\,\text{———}\right)$$

$$\Rightarrow \alpha_1\left(\operatorname{tg}\left(\frac{\phi^{\alpha_2}-\phi^{\alpha_1}}{4}\right) + \operatorname{tg}\left(\frac{\phi^{\alpha_2\alpha_1}-\phi}{4}\right)\right)$$
$$= \alpha_2\left(\operatorname{tg}\left(\frac{\phi^{\alpha_2}-\phi^{\alpha_1}}{4}\right) - \operatorname{tg}\left(\frac{\phi^{\alpha_2\alpha_1}-\phi}{4}\right)\right)$$

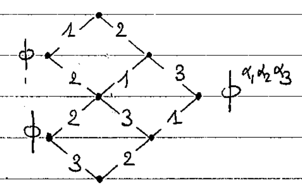
$$\Rightarrow (\alpha_1+\alpha_2)\operatorname{tg}\left(\frac{\phi^{\alpha_2\alpha_1}-\phi}{4}\right) = (\alpha_1-\alpha_2)\operatorname{tg}\left(\frac{\phi^{\alpha_1}-\phi^{\alpha_2}}{4}\right)$$

$$\Rightarrow \boxed{\phi^{\alpha_2\alpha_1} = \phi + 4\operatorname{arctg}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\operatorname{tg}\left(\frac{\phi^{\alpha_1}-\phi^{\alpha_2}}{4}\right)\right)}$$

and the same holds for the ϕ_t part of Bäcklund transf.

So, in this manner, we can actually construct (a) solution(s) and <u>never</u> have to solve the PDE of Bäcklund transf.

That is, for the third order, consider $\alpha_1, \alpha_2, \alpha_3$ and the lattice



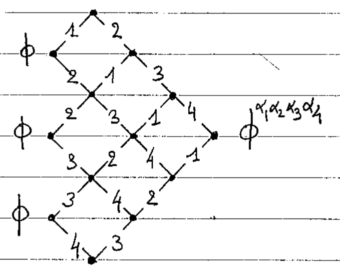
From the previous results:

$$\phi^{\alpha_1\alpha_2\alpha_3} = \phi^{\alpha_2} + 4\,\mathrm{arctg}\left(\frac{\alpha_1-\alpha_3}{\alpha_1+\alpha_3}\,\mathrm{tg}\left(\frac{\phi^{\alpha_1\alpha_2}-\phi^{\alpha_3\alpha_2}}{4}\right)\right)$$

$$= 4\,\mathrm{arctg}\left(C_2\, e^{(\alpha_2 t + l/\alpha_2 + u_2)/\rho}\right) \leftarrow \text{if the seed } \phi = 0$$

$$+ 4\,\mathrm{arctg}\left(\frac{\alpha_1-\alpha_3}{\alpha_1+\alpha_3}\,\mathrm{tg}\left\{ \mathrm{arctg}\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\,\mathrm{tg}\left(\frac{\phi^{\alpha_1}-\phi^{\alpha_2}}{4}\right)\right) - \mathrm{arctg}\left(\frac{\alpha_3-\alpha_2}{\alpha_3+\alpha_2}\,\mathrm{tg}\left(\frac{\phi^{\alpha_3}-\phi^{\alpha_2}}{4}\right)\right)\right\}\right)$$

For $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ one consider the lattice



etc.

Using the well known trigonometric identities of the tg, namely $\mathrm{tg}(a+b) = [\mathrm{tg}\,a + \mathrm{tg}\,b]/[1 - \mathrm{tg}(a)\mathrm{tg}(b)]$ and

$$\phi = 0$$

$$\phi^{\alpha_1} = 4\,\mathrm{arctg}\left(C_1\, e^{(\alpha_1 t + l/\alpha_1 + u_1)/\rho}\right)$$

$$\phi^{\alpha_2} = 4\,\mathrm{arctg}\left(C_2\, e^{(\alpha_2 t + l/\alpha_2 + u_2)/\rho}\right)$$

one gets

$$\phi^{\alpha_1\alpha_2} = 4\,\mathrm{arctg}\left(\left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)\times\left(\frac{C_1 e^{(\alpha_1 t + l/\alpha_1 + u_1)/\rho} - C_2 e^{(\alpha_2 t + l/\alpha_2 + u_2)/\rho}}{1 + C_1 C_2\, e^{((\alpha_1+\alpha_2)t + (1/\alpha_1 + 1/\alpha_2)l + u_1 + u_2)/\rho}}\right)\right)$$

For more material about this subject one can read:

- "Geometry of solitons", Chuu-Lian Terng & Karen Uhlenbeck, Notices of the AMS, volume 47, Number 1, pp. 17-25 (2000).
- "Bäcklund and Darboux Transformations, Geometry and modern applications in Soliton theory", C. Rogers & W.K. Schief, Cambridge University Press (2002)