Computing the Poisson kernel through Fourier transform (in the "I know the trick manner")

We seek $G(\vec{x})$ s.t. $\Delta G(\vec{x}) = \delta^3(\vec{x})$ (*)

Set $G(\vec{x}) = \int_{\mathbb{R}^3} \hat{G}(\vec{k})\, e^{-i\vec{k}.\vec{x}}\, d^3k$

$$\Leftrightarrow \quad \hat{G}(\vec{k}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} G(\vec{x})\, e^{+i\vec{k}.\vec{x}}\, d^3x$$

$$\delta^3(\vec{x}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i\vec{k}.\vec{x}}\, d^3k$$

pluging that in (*) yields

$$\Delta \int_{\mathbb{R}^3} \hat{G}(\vec{k}) e^{-i\vec{k}.\vec{x}}\, d^3k = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i\vec{k}.\vec{x}}\, d^3k$$

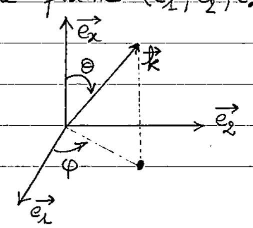
$$\hookrightarrow \int_{\mathbb{R}^3} (-i)^2 \|\vec{k}\|^2 \hat{G}(\vec{k})\, e^{-i\vec{k}.\vec{x}}\, d^3k = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{-i\vec{k}.\vec{x}}\, d^3k$$

stripping the integral and equating (that is up to null measure modifications) yields:

$$\hat{G}(\vec{k}) = \frac{-1}{(2\pi)^3} \times \frac{1}{\|\vec{k}\|^2} \quad \text{a.e.}$$

More importantly $G(\vec{x}) = \frac{-1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{1}{\|\vec{k}\|^2} e^{-i\vec{k}.\vec{x}}\, d^3k.$

Let us now choose a system of coordinates with respect to the frame $(\vec{e}_1, \vec{e}_2, \vec{e}_x)$



Using spherical coordinates yields, one sets $k = \|\vec{k}\|$,

$$G(\vec{x}) = \frac{-1}{(2\pi)^3}\int_0^{+\infty}\int_0^{\pi}\int_0^{2\pi} \frac{1}{\cancel{k^2}} e^{-ik\|\vec{x}\|\cos\theta}\, \cancel{k^2} \sin\theta\, d\varphi\, d\theta\, dk$$

$$= \frac{-1}{(2\pi)^2}\int_0^{+\infty}\int_0^{\pi} e^{-ik\|\vec{x}\|\cos\theta} \sin\theta\, d\theta\, dk$$

set, say, $u = -\cos\theta$, $du = \sin\theta\, d\theta$, $u|_{\theta=0} = -1$, $u|_{\theta=\pi} = 1$

$$\Rightarrow G(\vec{x}) = \frac{-1}{(2\pi)^2}\int_0^{+\infty}\int_{-1}^{1} e^{+ik\|\vec{x}\|u}\, du\, dk$$

$$= \frac{-1}{(2\pi)^2}\int_0^{+\infty} \frac{1}{ik\|\vec{x}\|}\left(e^{ik\|\vec{x}\|} - e^{-ik\|\vec{x}\|}\right) dk$$

$$= \frac{i}{(2\pi)^2} \times \frac{1}{\|\vec{x}\|}\int_0^{+\infty} \frac{1}{k}\left(e^{ik\|\vec{x}\|} - e^{-ik\|\vec{x}\|}\right) dk$$

Note that $\int_0^{+\infty} \frac{1}{k} e^{-ik\|\vec{x}\|}\, dk = \int_0^{-\infty} \frac{-1}{k} e^{+ik\|\vec{x}\|} (-dk) = -\int_{-\infty}^{0} \frac{1}{k} e^{ik\|\vec{x}\|}\, dk$

Meaning that $G(\vec{x}) = \frac{i}{(2\pi)^2} \times \frac{1}{\|\vec{x}\|}\int_{\mathbb{R}} \frac{1}{k} e^{ik\|\vec{x}\|}\, dk$

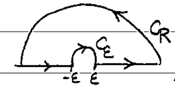
All is left to do is to compute the integral

$$I = \int_{\mathbb{R}} \frac{1}{k} e^{ikx}\, dk \qquad x = \|\vec{x}\| > 0$$

A solution is to write $I = \lim_{\varepsilon \to 0^+} I_\varepsilon = \lim_{\varepsilon \to 0^+} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{+\infty}\right) \frac{1}{k} e^{ikx}\, dk$

and one uses a complex analysis trick.

Consider $k \in \mathbb{C}$, note that $e^{ikx} = e^{i\,\mathrm{Re}(k)x}\, e^{-\mathrm{Im}(k)x}$

and consider the contour 

, no pole is enclosed by the contour $\Rightarrow$ the overall integral is null and one gets $0 = \curvearrowleft = \rightarrow \ \rightarrow + \frown_{\varepsilon} + \underbrace{\frown_{C_R}}_{0}$ (due to exponential decay)

The integral on C_R vanishes and one gets

$$I_{\varepsilon} = \rightarrow \ \rightarrow = -\frown_{\varepsilon} = +\frown_{\varepsilon}$$

meaning $I = \lim\limits_{\varepsilon \to 0^+} \int_{C_\varepsilon} \frac{1}{k} e^{ik\|\vec{x}\|} dk$, on C_ε one has $k = \varepsilon e^{i\theta}$, $0 \leq \theta \leq \pi$

$$= \lim_{\varepsilon \to 0^+} \int_0^{\pi} \frac{1}{\cancel{\varepsilon e^{i\theta}}} e^{i\varepsilon e^{i\theta}\|\vec{x}\|} \, i \cancel{\varepsilon e^{i\theta}} \, d\theta$$

$$= \lim_{\varepsilon \to 0^+} i \int_0^{\pi} e^{i\varepsilon e^{i\theta}\|\vec{x}\|} \, d\theta$$

$= i\pi$, thanks to the limit which kills higher order terms.

Finally one gets $\boxed{G(\vec{x}) = \frac{-1}{4\pi} \times \frac{1}{\|\vec{x}\|}}$

up to solutions of homogenous equation.

Remark that the integration over the contour $\rightarrow \smile \rightarrow$ yields the same result for $I = \int_{\mathbb{R}} \frac{1}{k} e^{ikx} dk$, this is not obvious, Green's functions are known to have various determinations (retarded/advanced solutions say).