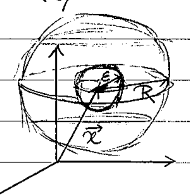
Determining the behavior of $G(\vec{x},\vec{x}')$ through the solution by a convolution

We would like to know if $\phi(\vec{x}) = \int_{\mathbb{R}^3} \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x'$ is a solution for the problem $\Delta\phi = \rho$.

Let us start with the integral $\int_{\mathbb{R}^3} \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x'$ which we view as the improper integral



$$\lim_{\substack{R\to+\infty \\ \varepsilon\to 0^+}} \int_{B_R\setminus B_\varepsilon} \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x'$$

with B_R and B_ε balls centered on $\vec{x}$ and of radius R and ε, respectively.

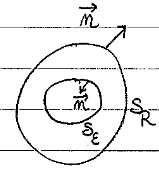
We choose to do it in that manner because some difficulties might be encountered at infinity and singularities of $G(\vec{x},\vec{x}')$ are expected at the coincidence limit $\vec{x}' \to \vec{x}$.

Now that we are on a safe ground suppose that there actually exists a solution ϕ of the problem $\Delta\phi = \rho$, then the integral reads:

$$\lim_{\substack{R\to+\infty \\ \varepsilon\to 0^+}} \int_{B_R\setminus B_\varepsilon} \overbrace{(\Delta\phi)(\vec{x}')}^{\rho(x')}\, G(\vec{x},\vec{x}')\, d^3x'$$

Applying Green formula thus yields

$$\lim_{\substack{R\to+\infty \\ \varepsilon\to 0^+}} \int_{B_R\setminus B_\varepsilon} (\Delta\phi)(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' = \lim_{\substack{R\to+\infty \\ \varepsilon\to 0^+}} \Big[\int_{B_R\setminus B_\varepsilon} \phi(\vec{x}')(\Delta G)(\vec{x},\vec{x}')\, d^3x'$$



$$+ \int_{S_R} \left\{ G(\vec{x},\vec{x}')(\vec{\nabla}\phi)(\vec{x}') - \phi(\vec{x}')(\vec{\nabla}G)(\vec{x},\vec{x}') \right\}\cdot\vec{n}\, dS$$

$$- \int_{S_\varepsilon} \left\{ G(\vec{x},\vec{x}')(\vec{\nabla}\phi)(\vec{x}') - \phi(\vec{x}')(\vec{\nabla}G)(\vec{x},\vec{x}') \right\}\cdot\vec{n}\, dS \Big]$$

Now it is asked that:

- ⊡ in $B_R\setminus B_\varepsilon$ $G(\vec{x},\vec{x}')$ is also a solution of the homogenous problem,
- ⊡ the integral on the outer boundary $(+\infty)$ vanishes

$$\Rightarrow G\nabla\phi\, dS \to 0 \quad \text{as} \quad R\to+\infty, \quad \|\vec{x}-\vec{x}'\| = R$$

since G is viewed as a solution too:

$$\xrightarrow{\text{grad.}}\ \Rightarrow \frac{1}{R}\phi^2 R^2 \to 0 \quad \text{as} \quad R\to+\infty$$

$$\Rightarrow \phi\sqrt{R} \to 0 \quad \text{as} \quad R\to+\infty$$

that is: solutions have to decay faster than $1/\sqrt{R}$.

- ⊡ at coincidence both limits have to hold

$$\lim_{\varepsilon\to 0} \varepsilon^2 G(\vec{x}, \vec{x}+\varepsilon\vec{n}) = 0$$

$$\lim_{\varepsilon\to 0} \varepsilon^2 \frac{\partial}{\partial r} G(\vec{x}, \vec{x}+r\vec{n})\Big|_{r=\varepsilon} = c \quad , \quad c \text{ finite value}$$

with $(\vec{\nabla} f)\cdot\vec{n}\big|_S = \frac{\partial f}{\partial r}$ on a sphere with r the radius with respect to the center.

Taking into account both of these limits pretty much say that

$$G(\vec{x},\vec{x}') \simeq \frac{-c}{r} + \dots \quad \text{for} \quad \|\vec{x}-\vec{x}'\| = r \ll \varepsilon$$

That is $\int_{\mathbb{R}^3} \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' = \lim_{\varepsilon\to 0} c \int_{S_\varepsilon} \phi(\vec{x}+\varepsilon\vec{n})\, d\Omega$

$$f(\vec{x}+\varepsilon\vec{n}) = f(\vec{x}) + \varepsilon\,\vec{n}\cdot(\vec{\nabla} f)(\vec{x}) + o(\varepsilon^2)$$

all the terms but the first are suppressed by $\lim_{\varepsilon\to 0}$ then

$$\int_{\mathbb{R}^3} \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' = 4\pi c\ \phi(\vec{x}')$$

and one chooses $c = \frac{1}{4\pi}$.

Now one might check that $G(\vec{x},\vec{x}') = \frac{-1}{4\pi} \times \frac{1}{\|\vec{x}-\vec{x}'\|}$ fulfill all of the above.

Generalization 1. $(\Delta+\lambda)\varphi = \rho$ on $\mathbb{R}^d$, $d \geq 3$

We get the behavior $\lim_{R\to+\infty} R^{\frac{d-2}{2}} G(\vec{x},\vec{x}') = 0$ (UV) $\|\vec{x}-\vec{x}'\| = R$

and $G(\vec{x},\vec{x}') = \frac{-1}{d-2} \times \frac{\Gamma(d/2)}{2\pi^{d/2}} \times \frac{1}{r^{d-2}} + \dots$ $\quad \|\vec{x}-\vec{x}'\| = r \ll 1$

$$= -\frac{\Gamma(\frac{d-2}{2})}{4\pi^{d/2}} \times \frac{1}{r^{d-2}} + \dots \quad \|\vec{x}-\vec{x}'\| = r \ll 1 \quad (IR)$$

It is worthy to notice that the asymptotic behaviors of $G(\vec{x},\vec{x}')$ are both independent of λ, while in between (in $B_R \setminus B_\varepsilon$) it obviously is dependent of λ ($\equiv$ mass term).

Generalization 2. $(\Delta+\lambda)\varphi = \rho$ on $\mathbb{R}^2$

We have $\lim_{R\to+\infty} G(\vec{x},\vec{x}') = 0$ as $\|\vec{x}-\vec{x}'\| = R$

and $\lim_{\varepsilon\to 0} \varepsilon G(\vec{x},\vec{x}') = 0$

$\lim_{\varepsilon\to 0} \varepsilon \frac{\partial}{\partial r} G(\vec{x},\vec{x}')\Big|_{r=\varepsilon} = c$ with $\|\vec{x}-\vec{x}'\| = \varepsilon \ll 1$.

implies $G(\vec{x},\vec{x}') = \frac{1}{2\pi} \ln r + \dots$ for $\|\vec{x}-\vec{x}'\| = \varepsilon \ll 1$.

If we seek radial solutions one gets $(\Delta+\lambda) f(r) = f'' + \frac{1}{r} f' + \lambda f = 0$ which is Bessel's equation of order zero whose solutions are $J_0(\sqrt{\lambda}\, r)$ and $Y_0(\sqrt{\lambda}\, r)$ but since

$$J_0(\sqrt{\lambda}\, r) = 1 + \dots$$

$$Y_0(\sqrt{\lambda}\, r) = \frac{2}{\pi} \ln\left(\frac{\sqrt{\lambda}\, r}{2}\right) + \dots \quad \text{h.o.t}$$

only Y_0 has the appropriate singularity as $r \to 0$.
Now

$$Y_0(\sqrt{\lambda}\, R) = \sqrt{\frac{2}{\pi R}} \sin\left(R - \frac{\pi}{4}\right) + \text{l.o.t}$$

thus for $\lambda > 0$: $\frac{1}{4} Y_0(\sqrt{\lambda}\, \|\vec{x}-\vec{x}'\|)$ fulfills all of the asked points.

For $\lambda = 0$, $\ln^2 r$ and $\ln r$ are two independent solutions none of which fulfills the appropriate behavior at $R \to +\infty$

Generalization 3. $(\Delta+\lambda)\varphi = \rho$ on $D \subset \mathbb{R}^d$, D bounded.

One gets the same structure of singularities as in $\mathbb{R}^d$ but for the integral on the boundary ∂D of D one asks that the solutions and the Green functions fulfill either

$$\varphi\,\big|_{\partial D} = 0 \quad \text{(Dirichlet)}$$

or $\quad (\vec{\nabla}\varphi).\vec{n}\,\big|_{\partial D} = 0 \quad$ (Neumann)

Conclusion. this trick is a good one as one gets insight on the needed asymptotic behavior of $G(\vec{x},\vec{x}')$ (at short and large distance) without solving the equation.
[this, however, doesn't assure us that there is one such, see $\Delta\varphi = 0$ on $\mathbb{R}^2$]