Übungen zur Vorlesung “Feldtheorie”

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Problem 1 Consider the length

$$\mu(p_0, p_1) = \int_0^1 \sqrt{g_{\mu\nu}(x)\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}\mathrm{d}\tau = \int_0^1 \sqrt{g_{\mu\nu}(x)\,\dot{x}^\mu\dot{x}^\nu}\mathrm{d}\tau \tag{1}$$

a) Write the Euler-Lagrange equations corresponding to (1) and show that they are solved for $x^\mu(\tau)$ a solution of

$$\ddot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = 0 \tag{2}$$

with

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\nu}\Big(\partial_\rho g_{\sigma\nu} + \partial_\sigma g_{\rho\nu} - \partial_\nu g_{\rho\sigma}\Big). \tag{3}$$

b) Show that the *covariant derivative* of a vector field

$$D_\rho v^\mu = \partial_\rho v^\mu + \Gamma^\mu_{\rho\sigma}v^\sigma \tag{4}$$

transforms as a tensor of type $\binom{1}{1}$ under a change of coordinate, with Γ given in (3). Hint. Such a tensor has to transform as $t^{\mu'}_{\nu'}(x') = \dfrac{\partial x'^{\mu'}}{\partial x^\mu}\,\dfrac{\partial x^\nu}{\partial x'^{\nu'}}t^\mu_\nu(x)$

c) For a tensor of arbitrary rank its derivative is given by

$$D_\rho T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} = \partial_\rho T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} + \Gamma^{\mu_1}_{\rho\sigma}T^{\sigma\mu_2\ldots\mu_p}_{\nu_1\ldots\nu_q} + \ldots - \Gamma^\sigma_{\rho\nu_1}T^{\mu_1\ldots\mu_p}_{\sigma\nu_2\ldots\nu_q} - \ldots \tag{5}$$

Show that for Γ given by (3) one has $D_\rho g_{\mu\nu} = 0$.

d) Compute $[D_\rho, D_\sigma]v^\mu$ and conclude that those derivatives do not commute in general.

e) Define the directional derivative along a vector (field) u as

$$D_u T^{\mu_1\cdots\mu_p}_{\nu_1\ldots\nu_q} = u^\rho D_\rho T^{\mu_1\cdots\mu_p}_{\nu_1\ldots\nu_q} \tag{6}$$

and check that then the geodesic equation (2) reads as

$$D_{\dot{x}}\,\dot{x}^\mu = 0. \tag{7}$$

Problem 2 *Some (useful) identities*
Show that for $g_{\mu\nu}$ diagonal the following identities hold:

$$\Gamma^\mu_{\nu\lambda} = 0, \qquad \Gamma^\mu_{\lambda\lambda} = -\frac{1}{2g_{\mu\mu}}\frac{\partial}{\partial x^\mu}g_{\lambda\lambda},$$

$$\Gamma^\mu_{\mu\lambda} = \frac{\partial}{\partial x^\lambda}\log\sqrt{|g_{\mu\mu}|}, \qquad \Gamma^\mu_{\mu\mu} = \frac{\partial}{\partial x^\mu}\log\sqrt{|g_{\mu\mu}|},$$

with $\mu \neq \nu \neq \lambda \neq \mu$ and no summation convention.

Problem 3 *Riemann normal coordinates*

Follow the general prescription (as in the lecture) to define the Riemann normal coordinates $\underline{z}(\underline{x})$ in the vicinity of a point $\underline{x}_0$, in the case of slow motion in a weak and slowly varying metric field (Newtonian approximation):

$$g_{\mu\nu}(\underline{x}) = \eta_{\mu\nu} + h_{\mu\nu}(\underline{x}),$$

where $|h| \ll 1$ and $|\partial_0 h| \ll |\partial_i h|$; and $|\vec{x} - \vec{x}_0| \ll |x^0 - x^0_0|$.

Show that, in this approximation,

$$\vec{z} = (\vec{x} - \vec{x}_0) - \vec{v}t - \frac{1}{2}\vec{a}t^2,$$

where $\phi(\underline{x}) = \frac{1}{2}c^2\, h_{00}(\underline{x})$, $v^i = h_{0i}(\underline{x}_0)c$, $\vec{a} = -\vec{\nabla}\phi(\underline{x}_0)$, and $t = z^0/c = (x^0 - x^0_0)/c$ up to $O(h)$. In other words, the generalized gravitational field is absent in an accelerated "free falling" system.