Übungen zur Vorlesung "Feldtheorie"

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Problem 1 *A wrong Theorem*

A function is called "harmonic", if $\underline{\Delta} f \equiv \vec{\nabla}^2 f = 0$.

"Theorem": Every harmonic function is constant.
"Proof": $0 = \int d^3x f \vec{\nabla}^2 f = -\int d^3x (\vec{\nabla} f)^2$ (by partial integration) implies $\vec{\nabla} f = 0$.

a) Show that $f(\vec{x}) = xy$ is a counter example.

b) What is wrong with the proof?

c) Show that the velocity field of an incompressible fluid ($\rho(t, \vec{x}) = const$) is the gradient of a harmonic function, whenever it has vanishing curl. Use the mass conservation $\partial_t \rho + \vec{\nabla}(\rho \vec{v}) = 0$.

d) Use the lesson of (b) to conclude that a nonzero velocity field in a container with boundary condition $\vec{v}|_B = 0$ must have nonzero curl.

Problem 2 *Reconstruction of a vector field from its divergence and curl*

Let $\vec{A}(\vec{x})$ be a vector field with divergence $f(\vec{x})$ and curl $\vec{g}(\vec{x})$. Define

$$\phi(\vec{x}) = -\int d^3x' \frac{f(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|} \qquad \text{and} \qquad \vec{C}(\vec{x}) = -\int d^3x' \frac{\vec{g}(\vec{x}')}{4\pi|\vec{x} - \vec{x}'|}.$$

a) How fast should $f(\vec{x})$ and $\vec{g}(\vec{x})$ decay with $r \equiv |\vec{x}| \to \infty$, so that the integrals are well-defined?

b) Show that $\vec{A}_1(\vec{x}) = -\vec{\nabla}\phi(\vec{x})$ has zero curl and divergence f.

c) Show that $\vec{A}_2(\vec{x}) = -\vec{\nabla} \times \vec{C}(\vec{x})$ has zero divergence and curl $\vec{g}$.

d) Is it true that $\vec{A} = \vec{A}_1 + \vec{A}_2$?